ON DERIVATIONS AND COMMUTATIVITY OF CERTAIN RINGS AND NEAR RINGS

THESIS

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By

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Dedicated

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Certificate

This is to certify that the thesis entitled **" ON DERIVATIONS AND COMMUTATIVITY OF CERTAIN RINGS AND NEAR RINGS"** is based on a part of the research work of **Ms. Rekha Rani** carried out under my guidance in the Department of Mathematics, Aligarh Muslim University, Aligarh. To the best of my Knowledge, the work included in the thesis is original and has not been submitted to any other University or Institution for the award of a degree.

It is further certified that **Ms. Rekha Rani** has fulfilled the prescribed conditions of duration and nature given in the statutes and ordinances of the Aligarh Muslim University, Aligarh.

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PREFACE

The present thesis entitled "**On derivations and commutativity of certain rings and near rings**" includes a part of the research work carried out by the author during the last five years at the Department of Mathematics, Aligarh Muslim University, Aligarh. The thesis comprises five chapters and each chapter is subdivided into various sections. The definitions, examples, results and remarks etc. have been specified with double decimal numbers. The first figure denotes the chapter, the second represents the section in the chapter and the third points out the number of the definition, the example, the result or the remark as the case may be, in a particular chapter. For example, Theorem 4.2.3 refers to the third theorem appearing in the second section of the fourth chapter.

Chapter 1 of the thesis contains some preliminary notions, basic definitions and important well-known results which may be needed for the development of the subsequent text. This chapter as a matter of fact, aims at making the present thesis as self contained as possible. However, the basic knowledge of the ring theory has been preassumed and no attempt is made to include the proofs of the known results presented in this chapter.

In a remarkable application to the ring theory in its infancy, J. H. M. Wedderburn discovered in 1905 that every finite division ring is necessarily commutative. Besides its own intrinsic beauty and striking applications, the Wedderburn theorem has served as a jumping-off point for investigating commutativity of rings under many functional constraints which has attracted a wide circle of mathematicians like Jacobson, Faith, Amitsur, Herstein, Bell, Ligh, Tominaga, Abujabal and Quadri. In the second chapter, we have generalized some already known results on commutativity of rings. One of our objectives of the chapter is to present some useful and refreshing techniques which make the proofs easier and shorter. In section 2.2, we extend a theorem of Kaplansky which asserts that a semisimple ring in which power of each element is central, must be commutative. Earlier, this theorem was generalized by many algebraists including Faith [62], Lithman [101], Herstein [72] and Bell [30] in different directions but most of the results were restricted to some special classes of rings. We have succeeded in generalizing the theorem of Kaplansky for the ring with unity imposing a torsion condition on the ring elements [cf. Theorem 2.2.1]. The result has been established using a reduction formula due to J. Tong [120]. One of the early extensions of Kaplansky Theorem due to I. N. Herstein was concerned to a ring having no nonzero nil ideal. This result of Herstein has been further extended in section 2.3 imposing an additional condition on the underlying ring and its proof is based on Jacobson structure theory of rings. In the last section, we prove a result using W. Streb's classification of noncommutative rings. Our theorem of this section, in fact, extends the results of many mathematicians including that of Jacobson [88] which, in turn, generalizes the classical theorem of Wedderburn [124] mentioned above and the result that a Boolean ring (satisfying $x^2 = x$) is necessarily commutative.

A ring element x for which there exists a positive integer $n > 1$ such that $x^n = x$ is said to be potent element and a ring in which every element is potent may be called J-ring (after the name of famous algebraist Nathan Jacobson). Evidently, J-rings are generalized Boolean rings. During the last few decades many algebraists obtained the direct sum decomposition of the rings satisfying various generalized Boolean conditions ; to mention a few ; Ligh and Luh [100], Bell and Ligh [38] and Quadri et. al [114]. Chapter 3 has been devoted to the similar study. In section 3.2, we obtain a decomposition of rings satisfying any one of the following conditions: (P_1) $xy = (xy)^n p(x, y)$, (P_2) $xy = x^m y^n p(x, y)$, (P_3) $xy = y^m x^n p(x, y)$, (P_4) $xy = x^m p(x, y)x^n$ and $xy = y^m p(x, y)y^n$, for all pairs of ring elements x and y, where $p(X, Y) \in \mathbb{Z}[X, Y]$, the ring of polynomials over the ring $\mathbb{Z}$ of integers. In section 3.3, the structure of rings is discussed, in case the mentioned conditions are satisfied for some restricted elements of the rings.

Having observed that the analogous hypotheses of the theorems of chapter 3 do not quite yield the direct sum decompositions for near rings (cf. Example 4.2.1), we define in Chapter 4, a weaker notion of orthogonal sum: *A near ring R is an orthogonal sum $(A \uplus B)$ of subnear rings A and B if $AB = BA = \{0\}$ and each element of R has a unique representation in the form $a + b$ with $a \in A$ and $b \in B$.* In section 4.2, we obtain an orthogonal sum decomposition for a $d - g$ near ring R satisfying any one of the following conditions : (I) $xy = (xy)^n p(xy)$, (II) $xy = x^m y^n p(xy)$, (III) $xy = y^m x^n p(xy)$, (IV) $xy = x^m p(xy)x^n$ and (V) $xy = y^m p(xy)y^n$, (I^*) $xy = (xy)^n p(yx)$, (II^*) $xy = x^m y^n p(yx)$, (III^*) $xy = y^m x^n p(yx)$, (IV^*) $xy = x^m p(yx)x^n$ and (V^*) $xy = y^m p(yx)y^n$, where $p(xy)$ denotes an element of R which is a finite sum of powers $(xy)^k$, for $k \geq 2$ and additive inverses of such powers. In the subsequent section, we attempt to extend the results of section 4.2 for D-near rings, a wider class than that of $d - g$ near rings. In the last section, we establish that a $d - g$ near ring satisfying any one of the above conditions turns out to be a commutative ring if either $R^2 = R$ or R contains unity.

Recently, there has been an ongoing interest among algebraists (particularly ring theorists) in studying commutativity of prime and semiprime rings admitting some special types of maps such as Lie and

Jordan maps, automorphisms, commutativity preserving maps and derivations. Chapter 5, the last one of our thesis addresses to this type of work viz-a-viz what we call "Generalized Derivation" (Definition 5.3.1). In section 5.2, some basic definitions and preliminary results are included which help develop the subsequent text. Unlike known results of chapter 1, we prefer to give the outlines of the proofs of the results presented in this section in order to familiarize the reader with the techniques generally used in case of derivations and that's why we designate them as **Propositions**. In section 5.3, we obtain the commutativity of a prime ring R admitting a generalized derivation F with associated derivation d satisfying any one of the following conditions : (i) $[d(x), F(y)] = [x, y]$, (ii) $[d(x), F(y)] + [x, y] = 0$, (iii) $[d(x), F(y)] = 0$, (iv) $d(x) \circ F(y) = 0$, (v) $d(x) \circ F(y) = x \circ y$, (vi) $d(x) \circ F(y) + x \circ y = 0$, (vii) $d(x)F(y) - xy \in Z(R)$ and (viii) $d(x)F(y) + xy \in Z(R)$, for all x, y in some appropriate subset of the ring R. In the last section, a result of Ashraf and Nadeem [14] about ordinary derivation has been extended for generalized derivation in the setting of Lie ideal. In fact, we prove the following : Let R be a prime ring and U be a nonzero Lie ideal of R with $u^2 \in U$, for all $u \in U$. If R admits a generalized derivation F with associated derivation d satisfying any one of the following conditions : (i^*) $F(uv) + uv \in Z(R)$, (ii^*) $F(uv) - uv \in Z(R)$, (iii^*) $F(uv) + vu \in Z(R)$ and (iv^*) $F(uv) - vu \in Z(R)$, for all $u, v \in U$. Then U must be central.

At places, examples are provided to justify the conditions imposed on the hypotheses of various results. The extensions of some of the results presented in the exposition may not be outrightly ruled out but choice of our examples shows that they can not be generalized arbitrarily. Also suitable remarks are given sometime to explain the theory and sometime to conjecture the extensions of the results.

In the end, an exhaustive bibliography of the existing material related to the subject matter of our thesis is included which may serve as source material for those, interested in the domain of our research.

Two papers of the author related to some portions of Chapter 4 have already been published in Radovi Mathematicki, Vol. 12, No. 1 (2003) and Acta. Sci. Natur. Univ. Jilin, Vol. 41, No. 3 (2003), whereas two papers based on the material of Chapter 5 have been accepted for publication in Southeast Asian Bull. Math. and Trends in Theory of Rings and Modules, Anamaya Publishers, New Delhi, 2004. Several papers related to material of other chapters are in the process of acceptance.

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Rekha Rani

(Rekha Rani)

Chapter-1

PRELIMINARIES

The present chapter is devoted to review some basic notions, important terminology and known results with a view to making our thesis as self contained as possible. Of course, the elementry knowledge of the algebraic concepts like groups, rings, ideals, fields, and homomorphisms etc. has been pre-assumed and no attempt has been made to discuss them here. Suitable examples and necessary remarks are given at proper places.

1.2 SOME RING THEORETIC CONCEPTS

In the present section, we give a brief exposition of some important terminology in Ring theory. Throughout, until otherwise specified R represents an associative ring (may be without unity and need not be commutative). For any pair of elements $x, y \in R$, the commutator $xy - yx$ will be denoted by $[x, y]$ and anti-commutator $xy + yx$ by $x \circ y$.

Definition 1.2.1 (Characteristic of a Ring). The least positive integer n (if exists) such that $nx = 0$, for all $x \in R$ is called the characteristic of the ring R which is generally expressed as char$R = n$. If no such positive integer exists, then R is said to have characteristic zero.

Remark 1.2.1. Obviously, if char$R \neq m$, then for some $x \in R$, $mx = 0$ implies that $x = 0$.

Definition 1.2.2 (Torsion Free Element). An element $x \in R$ is said to be n-torsion free if $nx = 0$ implies that $x = 0$. If $nx = 0$ implies $x = 0$, for every

$x \in R$, then we say that R is n-torsion free.

Definition 1.2.3 (Idempotent Element). An element e of a ring R is said to be idempotent if $e^2 = e$.

Remark 1.2.2. It is trivial that zero of a ring R is an idempotent element. Moreover, if R contains unity 1, then 1 is also idempotent. However, there may exist many idempotent elements in R other than 0 and 1.

Definition 1.2.4 (Nilpotent Element). An element x of a ring R is said to be nilpotent if there exists a positive integer n such that $x^n = 0$.

Remark 1.2.3. It is trivial that zero of a ring R is nilpotent. Moreover, every nilpotent element is necessarily a divisor of zero. For if $x \neq 0$ and n is the smallest positive integer such that $x^n = 0$, then $n > 1$ and $x(x^{n-1}) = 0$ with $x^{n-1} \neq 0$.

Definition 1.2.5 (Centre of a Ring). The centre $Z(R)$ of a ring R is the collection of all those elements of R which commute with each element of R, that is,

$$Z(R) \;=\; \{x \in R \mid xy = yx, \text{ for all } y \in R\}.$$

Definition 1.2.6 (Centralizer). Let S be a non-void subset of a ring R. Then the centralizer $C_R(S)$ of S in R is defind as

$$C_R(S) = \{x \in R \mid xs = sx, \text{ for all } s \in S\}.$$

Definition 1.2.7 (Nilpotent Ideal). A right (left, two sided) ideal I of a ring R is said to be nilpotent if there exists a positive integer $n > 1$

such that $I^n = \{0\}$.

Definition 1.2.8 (Nil Ideal). A right (left, two sided) ideal I of a ring R is said to be nil if each of its element is nilpotent.

Example 1.2.1. Let $R=\left\{\begin{pmatrix} a & 0 \\ b & c \end{pmatrix} ; a, b, c \in \mathbb{Z}\right\}$. Let I be an ideal of R generated by $\begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix}$. Then I is nilpotent and also a nil ideal.

Remark 1.2.4.

(*i*) If every element of a ring R is nilpotent, then R itself is called a nil ring.

(*ii*) Every nilpotent ideal is nil but a nil ideal need not be necessarily nilpotent.

Example 1.2.2. Let p be a fixed prime and for each positive integer i, let R_i be the ideal in $\mathbb{Z}/(p^{i+1})$, consisting of all nilpotent elements of $\mathbb{Z}/(p^{i+1})$, that is, consisting of the residue classes modulo p^{i+1} which contain multiples of p. Then $R_i^{i+1} = (0)$, whereas $R_i^k \neq (0)$, for $k < i+1$. Now consider the discrete direct sum T of the rings R_i (i=1, 2, 3,.......). Since each element of T differs from zero in only a finite number of components i.e., each element of T is nilpotent. Then T is a nil ideal in T but not a nilpotent ideal.

Definition 1.2.9 (Commutator Ideal). The commutator ideal $C(R)$ of a ring R is the ideal generated by all commutators $[x, y]$ with $x, y \in R$.

Definition 1.2.10 (Prime Ideal). An ideal P of a ring R is said to be a prime ideal if and only if it has the property that for any two ideals A, B in R, whenever $AB \subseteq P$, then $A \subseteq P$ or $B \subseteq P$.

Remark 1.2.5. Equivalently, an ideal P in a ring R is prime if and only if any one of the following holds :

(*i*) If for any $a, b \in R$ such that $aRb \subseteq P$, then $a \in P$ or $b \in P$.

(*ii*) If (a) and (b) are principal ideals in R such that $(a)(b) \subseteq P$, then $a \in P$ or $b \in P$.

(*iii*) If R is a commutative ring such that for any $a, b \in R$, $ab \in P$, then $a \in P$ or $b \in P$.

(*iv*) If U and V are right (left) ideals in R such that $UV \subseteq P$, then $U \subseteq P$ or $V \subseteq P$.

Definition 1.2.11 (Semiprime Ideal). An ideal I in a ring R is said to be a semiprime ideal if for any ideal A in R , whenever $A^2 \subseteq I$, then $A \subseteq I$.

Remark 1.2.6.

(*i*) A prime ideal is necessarily semiprime but the converse need not be true in general.

(*ii*) Intersection of prime (semiprime) ideals is semiprime. Thus in the ring $\mathbb{Z}$ of integers, ideal $(2) \cap (3) = (6)$ is semiprime which is not prime.

Definition 1.2.12 (Maximal Ideal). An ideal M of a ring R is called maximal, if

(*i*) $M \neq R$,

(*ii*) there exists no ideal J in R such that $M \subset J \subset R$.

Remark 1.2.7.

(*i*) If $M \neq R$ is a maximal in R, then for any ideal J of R, $M \subseteq J \subseteq R$ holds only when either $J = M$ or $J = R$.

(*ii*) Every maximal ideal in a commutative ring is a prime ideal.

Definition 1.2.13 (Jacobson Radical). The Jacobson radical $J(R)$ of a ring R is the intersection of all maximal left (right) ideals of R.

Definition 1.2.14 (Annihilator). If M is a subset of a commutative ring R, then the annihilator of M, denoted by $Ann(M)$ is the set of all elements r of R such that $rm = 0$, for all $m \in M$. Thus

$$Ann(M) = \{r \in R \mid rm = 0, \text{ for all } m \in M\}.$$

Definition 1.2.15 (Prime Ring). A ring R is said to be prime if and only if zero ideal (0) is prime ideal in R.

Remark 1.2.8. Equivalently, a ring R is prime if and only if any one of the following holds :

(*i*) If (a) and (b) are principal ideals in R such that $(a)(b) = 0$, then $a = 0$ or $b = 0$.

(*ii*) If $a, b \in R$ such that $aRb = (0)$, then $a = 0$ or $b = 0$.

Definition 1.2.16 (Semiprime Ring). A ring R is said to be semiprime if it has no nonzero nilpotent ideals.

Remark 1.2.9. Equivalently, a ring R is prime if and only if

(*i*) For any $x \in R$, whenever $xRx = \{0\}$, then $x = 0$.

(*ii*) The centre of a semiprime ring contains no nonzero nilpotent elements.

(*iii*) In a semiprime ring R, the centre of a nonzero one sided ideal is contained in the centre of R. In particular, any commutative one sided ideal is contained in the centre of R.

Definition 1.2.17 (Simple Ring). A ring R with more than one element is said to be simple if its only ideals are the two trivial ideals, namely, (0) and R itself.

Definition 1.2.18 (Semisimple Ring). A ring R is said to be semisimple if its Jacobson radical is zero.

Definition 1.2.19 (Boolean Ring). A ring R is said to be Boolean if all of its elements are idempotent i.e., $x^2 = x$, for all $x \in R$.

Definition 1.2.20 (Direct Sum and Subdirect Sum of Rings). Let S_i, $i \in U$ be a family of rings indexed by the set U and S denote the set of all functions defined on the set U such that for each $i \in U$, the value of function at i is an element of S_i. If addition and multiplication in S are defined as : $(a+b)(i) = a(i) + b(i)$, $(ab) = a(i)b(i)$, for all $a, b \in S$, then S is a ring which is called the complete direct sum of the rings S_i, $i \in U$. The set of all functions in S which take the value zero at all but at most a finite number of elements $i \in U$ is a subring of S which is called the discrete direct sum of rings S_i, $i \in U$. However, if U is a finite set, the complete (discrete) direct sum of rings S_i, $i \in U$, as defined above is called a direct sum of rings S_i, $i \in U$.

Let T be a subring of the direct sum S of rings S_i and for each $i \in U$ let $\theta_i \in U$ be a homomorphism of S onto S_i defined by $a\theta_i = a(i)$, for $a \in S$. If $T\theta_i = S_i$ for every $i \in U$, then T is said to be a subdirect sum of the ring S_i, $i \in U$.

Definition 1.2.21 (Lie and Jordan Structure). Let R be an associative ring. We can induce on R two new operations as follows :

(i) For all $x, y \in R$, the Lie product $[x, y] = xy - yx$.

(ii) For all $x, y \in R$, the Jordan product $x \circ y = xy + yx$.

The additive group $(R, +)$ together with the Lie product (resp. Jordan product) is sometimes called Lie (resp. Jordan) ring.

Remark 1.2.10. For any $x, y, z \in R$, the following identities hold :

(i) $[xy, z] = x[y, z] + [x, z]y$.

(ii) $[x, yz] = y[x, z] + [x, y]z$.

(iii) $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$. This identity is generally known as Jacobi identity.

(iv) $x \circ (yz) = (x \circ y)z - y[x, z] = y(x \circ z) + [x, y]z$.

(v) $(xy) \circ z = x(y \circ z) - [x, z]y = (x \circ z)y + x[y, z]$.

Definition 1.2.22 (Lie (Jordan) Subring). A nonvoid subset U of a ring R is Lie (resp. Jordan) subring of R if U is an additive subgroup of R and $a, b \in U$ implies that $[a, b]$ (resp. $(a \circ b)$) is also in U.

Definition 1.2.23 (Lie (Jordan) Ideal). An additive subgroup $U \subset R$ is said to be a Lie (resp. Jordan) ideal of R if whenever $u \in U$ and $r \in R$, then $[u, r]$ (resp.$(u \circ r)$) is also in U.

Example 1.2.3. Let $R = \left\{ \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \mid a, b, c \in \mathbb{Z}_2 \right\}$. Then it can be easily seen that $U = \left\{ \begin{pmatrix} a & 0 \\ b & a \end{pmatrix} \mid a, b \in \mathbb{Z}_2 \right\}$ is a Lie ideal of R and $U = \left\{ \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix} \mid b \in \mathbb{Z}_2 \right\}$ is a Jordan ideal of R.

Definition 1.2.24 (Commuting Function). Let S be a subset of R. A function $F : R \longrightarrow R$ is said to be a commuting function on S if $[F(x), x] = 0$, for all $x \in S$.

Definition 1.2.25 (Centralizing Function). Let S be a subset of R. A function $F : R \longrightarrow R$ is said to be a centralizing function on S if $[F(x), x] \in Z(R)$, for all $x \in S$ i.e., $[[F(x), x], z] = 0$, for all $x \in S$ and $z \in R$.

1.3 NEAR RINGS

This section deals with some preliminary concepts and simple properties of near rings.

Definition 1.3.1 (Near Ring). A left near ring R is a triple $(R, +, *)$ with two binary operations $+$ and $*$ such that

(i) $(R, +)$ is a group (not necessarily abelian).

(ii) $(R, *)$ is a semigroup.

(iii) $a * (b + c) = a * b + a * c$, for all $a, b, c \in R$.

Analogously, if instead of (iii), we have the right distributive low

$(iii)'$ $(a + b) * c = a * c + b * c$

holds, then R is said to be a right near ring.

As in both the cases, the theory of near rings runs completely parrallel, we may consider left near rings throughout and for simplicity call them as near rings.

Example 1.3.1. (*i*) The set of all identity preserving mappings acting on the right of an additive group G (not necessarily abelian) into itself with pointwise addition and composition of mappings as multiplication is the most natural example of a right near ring.

(*ii*) $R = \{0, a\}$ with addition and multiplication defined as follows :

+	0	a
0	0	a
a	a	0

*	0	a
0	0	a
a	0	a

It is easily checked that R is a left near ring.

(*iii*) For more examples one may consult [54].

Definition 1.3.2 (Distributive Element). An element x of a near ring R is said to be distributive if $(y + z)x = yx + zx$, for all $y, z \in R$.

Definition 1.3.3 (Distributive Near Ring). A near ring R is said to be distributive if all of its elements are distributive.

Remark 1.3.1. In any near ring R,

(*i*) $x0 = 0$, for all $x \in R$, but not necessarily $0x = 0$. However, if R is distributive, then $0x = 0$.

(*ii*) $x(-y) = -xy$, for all $x, y \in R$, but not necessarily $(-x)y = -xy$. However, if R is distributive, then $(-x)y = -xy$.

Definition 1.3.4 (Additive Centre). An additive centre of a near ring R is the set of all those elements of R which commute with every element of R under addition.

Multiplicative centre of a near ring is defined in the same manner as we have defined in the case of rings (cf. Definition 1.2.5).

Definition 1.3.5 (Distributively Generated Near Ring). A near ring R is said to be distributively generated $(d-g)$, if it contains a multiplicative subsemigroup of distributive elements which generates the additive group $(R,+)$.

Example 1.3.2. The near ring generated additively by all the endomorphisms of a group $(G,+)$ (not necessarily abelian), is a distributively generated near ring.

Definition 1.3.6 (Ideal). An ideal of a near ring R is defined to be a normal subgroup I of R^+ such that

(i) $RI \subseteq I$.

(ii) $(x+i)y - xy \in I$, for all $x, y \in R$ and $i \in I$.

Normal subgroup of $(R,+)$ satisfying (i) are called the left ideals and satisfying (ii) are called right ideals.

In case of a $d-g$ near ring, the condition (ii) above may be replaced by

$(ii)^*$ $IR \subseteq I$.

Definition 1.3.7 (Near Ring Homomorphism). A mapping $f : R \longrightarrow R^*$ of a near ring R into another near ring R^* is called a near ring homomorphism if $f(x+y) = f(x) + f(y)$ and $f(xy) = f(x)f(y)$, for all $x, y \in R$.

Definition 1.3.8 (Zero-symmetric Near Ring). A near-ring R is said to be zero-symmetric, if $0x = 0$, for all $x \in R$ (recall that left distributivity yields $x0 = 0$).

Example 1.3.3. Let $R = \{0, a, b, c\}$ with addition and multiplication tables defined as below :

+	0	a	b	c
0	0	a	b	c
a	a	b	c	0
b	b	c	0	a
c	c	0	a	b

*	0	a	b	c
0	0	0	0	0
a	0	c	b	a
b	0	0	0	0
c	0	a	b	c

It can be easily verified that R is a zero-symmetric near ring.

Remark 1.3.2. A $d-g$ near ring is necessarily zero-symmetric.

Definition 1.3.9. (Zero-commutative Near Ring). A near ring R is said to be zero-commutative, if $xy = 0$ implies that $yx = 0$, for all $x, y \in R$.

Example 1.3.4. $R = \{0, a, b, c\}$ with addition and multiplication tables defined as below :

+	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

*	0	a	b	c
0	0	0	0	0
a	0	a	a	0
b	0	b	b	0
c	0	0	0	0

Then $(R, +, *)$ is a zero-commutative near ring.

1.4 SOME WELL KNOWN RESULTS

In this section, we state some well-known results which will be frequently used in the development of the subsequent chapters.

Theorem 1.4.1 (Daif and Bell [56]). Let R be a semiprime ring and I be a nonzero ideal of R. If a in R centralizes the set $[I, I]$, then a centralizes I.

Theorem 1.4.2 (Jacobson [87]). Let R be a ring in which for every $x \in R$ there exists an integer $n = n(x) > 1$, depending on x such that $x^{n(x)} = x$, then R is commutative.

Theorem 1.4.3 (Frohlic [65]). A $(d - g)$ near ring R with unity 1 is a ring if $(R, +)$ is abelian or R is distributive.

Theorem 1.4.4 (Bell [22]). Let R be a zero-symmetric near ring having no nonzero nilpotent elements. Then

(i) every distributive idempotent is central;

(ii) for every idempotent e and every element $x \in R$, $ex^2 = (ex)^2$;

(iii) if R has a multiplicative identity element, then all idempotents are central.

Theorem 1.4.5 (Neumann [108]). The additive group of a division near ring is abelian.

Chapter-2

ON SOME CLASSICAL COMMUTATIVITY THEOREMS

2.1 INTRODUCTION

During the second half of the last century, a great deal of research was done which showed that certain conditions when imposed on a given ring, render it commutative. In the present chapter, we carry on this type of investigation further and extend some previously obtained results. Most of the early research workers like N. Jacobson, I. N. Herstein, N. H. McCoy, C. Faith and I. Kaplansky established their results using general structure theory of rings. Their proofs were complicated and lengthy. One of our objects in this chapter is also to present some refreshing tools for proving our results, besides extending the known theorems.

In section 2.2, we extend a theorem of Kaplansky which asserts that a semisimple ring in which power of each element is central, must be commutative. This theorem has been generalized by many authors including Faith [62], Lithman [101], Herstein [72] and Bell [30] in different directions but most of the results were restricted to some special classes of rings. We have succeeded in generalizing the theorem of Kaplansky for the ring with unity imposing a torsion condition on the ring. The result has been established using a reduction formula due to J. Tong [120].

One of the early extensions of Kaplansky Theorem due to I. N. Herstein was concerned to a ring having no nonzero nil ideal. This result of Herstein has been further extended in the subsequent section imposing an additional condition on the underlying ring. The proof of the theorem of this section is based on Jacobson structure theory of rings.

In the last section, we prove a result using W. Streb's classification of noncommutative rings. Our theorem of this section, in fact, extends the results of many research workers including that of Jacobson [88] which in turn generalizes the classical theorem of Wedderburn [124] that a finite division ring must be commutative and the result that a Boolean ring (satisfying $x^2 = x$, for all ring elements x) is necessarily commutative.

2.2 AN EXTENSION OF KAPLANSKY THEOREM

We know that *a ring is said to be commutative if and only if* $[x, y] = 0$, *for every pair* x, y *of ring elements.* This definition of commutativity of a ring prompts us to investigate the commutativity of a ring if there exists a positive integer n larger than 1 such that $[x^n, y] = 0$, for all pairs x, y of the ring elements. The noncommutative ring of 3×3 strictly upper triangular matrices over the ring $\mathbb{Z}$ of integers rules out the possibility of arbitrary rings with $[x^n, y] = 0$ to be commutative. Despite such bad examples, algebraists have been investigating the classes of rings which turn out to be commutative under the mentioned condition. One of the early results proved in this direction was due to Kaplansky [89] :

Theorem K. Let R be a semisimple ring in which there exists a positive integer $n \geq 1$ such that $[x^n, y] = 0$, for all $x, y \in R$. Then R must be commutative.

The above result attracted many algebraists including Carl Faith and

I. N. Herstein. However, most of the results available in the literature are about very restricted classes of rings. For example, Faith [62] established the result for division rings whereas Herstein [72] proved commutativity of rings in which commutator ideal is not nil. In the present section, we extend Theorem K for ring with unity 1, imposing torsion condition on the elements of the ring. In fact, we establish the following :

Theorem 2.2.1. Let R be a ring with unity 1 in which there exists a pair of positive integers $m \geq 1$, $n \geq 1$ such that $[x^m, y^n] = 0$, for all $x, y \in R$. If R is $m!n!$ torsion free, then R is necessarily commutative.

The following lemma is essentially proved in [120] will be extensively used in proving our theorem.

Lemma 2.2.1. Suppose R is an associative ring with unity 1. For any $x \in R$, set

$$S_0^r(x) = x^r$$

and

$$S_k^r(x) = S_{k-1}^r(1+x) - S_{k-1}^r(x), \ k \geq 1.$$

Then

(i) $S_{r-1}^r(x) = r![\frac{1}{2}(r-1) + x]$

(ii) $S_r^r(x) = r!$

(iii) $S_j^r(x) = 0$, for $j > r$.

Proof of Theorem 2.2.1. Using the notations of the above lemma, the condition of our theorem can be written as follows :

(2.2.1) $$[S_0^m(x), y^n] = 0, \text{ for all } x, y \in R.$$

On replacing x by $1 + x$ in the above identity, we have

$$[S_0^m(1 + x), y^n] = 0, \text{ for all } x, y \in R.$$

That is,

$$[S_1^m(x) + S_0^m(x), y^n] = 0, \text{ for all } x, y \in R.$$

Since commutator function is linear in both the coordinates, we have

$$[S_1^m(x), y^n] + [S_0^m(x), y^n] = 0.$$

In view of (2.2.1), this yields

$$[S_1^m(x), y^n] = 0, \text{ for all } x, y \in R.$$

Again replace x by $1 + x$ to get,

$$[S_1^m(1 + x), y^n] = 0, \text{ for all } x, y \in R.$$

Now repeating the process $(m - 1)$-times and using the Lemma 2.2.1, we obtain

$$[S_{m-1}^m(x), y^n] = 0, \text{ for all } x, y \in R.$$

Thus by (ii) of Lemma 2.2.1, we get

$$[\tfrac{1}{2}(m - 1)m! + m!x, \ y^n] = 0,$$

i.e.,

(2.2.2) $$m![x, y^n] = 0, \text{ for all } x, y \in R.$$

Now once again writing the above relation in the notations of Lemma 2.2.1, we have

$$m![x, S_0^n(y)] = 0, \text{ for all } x, y \in R.$$

This time working in the second coordinate of the commutator and proceeding in the same way as above, we finally get

$$m!n![x, y] = 0 \text{ for all } x, y \in R.$$

Since R is $m!n!$ torsion free, we have $[x, y] = 0$, which shows that R is commutative. □

Remark 2.2.1. Evidently, for $m = 1$ or $n = 1$, the above theorem turns out to be an extension of Theorem K due to Kaplansky. Our theorem also includes the theorems of Faith [62], Lithman [101] and many others.

Remark 2.2.2. A cursory look at the proof of the above theorem will reveal that the result remains still valid if the ring under consideration is $m!$ as well as $n!$ torsion free. Also the condition on the hypothesis can be further weakend by assuming that the commutators in R are $m!$ and $n!$ torsion free.

Remark 2.2.3. The following example demonstrates that the torsion condition on the commutators of the ring of our theorem can not be dropped.

Example 2.2.1. Consider the ring

$$R=\left\{ aI_3 + D_0 \mid D_0 = \begin{pmatrix} 0 & b & c \\ 0 & 0 & d \\ 0 & 0 & 0 \end{pmatrix} \; ; \; I_3 \text{ is } 3 \times 3 \text{ identity matrix and } a, b, c, d \in GF(2) \right\}.$$

It is readily verified that R is a noncommutative ring with unity I_3 satisfying $[x^2, y] = 0$, for all $x, y \in R$. Indeed, R is not 2-torsion free.

2.3 AN EXTENSION OF A THEOREM OF HERSTEIN

In a paper [79], I. N. Herstein introduced the concept of the hypercentre of a ring.

Definition 2.3.1 (Hypercentre). The hypercentre $T(R)$ of a ring R is the totality of all those elements of R which commute with some power of each element in R, the power may be localized in the sense that it may depend on the elements. Thus

$$T(R) = \{r \in R \mid rx^n = x^n r, \text{ where } n = n(r, x) \text{ is a positive integer}\}.$$

Trivially, $Z(R) \subseteq T(R)$. There exist enough noncommutative nil rings to show that in general $T(R)$ need not concide with $Z(R)$. In the previous section, we have seen that if R is a ring with unity in which there exists a positive integer n such that $rx^n = x^n r$, for all r, x in R and elements of R are well-behaved in the sense that the ring is $n!$-torsion free, then R is commutative and so $T(R) = Z(R)$. In another paper [81], Herstein proved the following :

Theorem H. Let R be a ring in which, given $x, y \in R$, there exist integers $m = m(x, y) \geq 1$, $n = n(x, y) \geq 1$ such that $[x^m, y^n] = 0$. If in addition, R has no nonzero nil ideals, then R must be commutative.

Motivated by these observations, one may conjecture that instead of torsion condition or absence of nil commutator ideal, some other constraints on the elements of R should also turn the ring commutative. Working on these lines, we extend Theorem H as follows :

Theorem 2.3.2. Let R be a ring with unity 1 in which there exist positive integers m and n satisfying

(C_1) : $[x^m, y^n] = 0$, for all $x, y \in R$.

(C_2) : $(xy)^n = (yx)^n$, for all $x, y \in R$.

If in addition, integers m and n are relatively prime, then R must be commutative.

The proof of the following lemma can be found in [88].

Lemma 2.3.2. If $[x, [x, y]] = 0$, for all $x, y \in R$, then $[x^n, y] = nx^{n-1}[x, y]$ holds for every positive integer n.

The following lemma is also available in existing literature. However, we reprove it here in a general setting. Also our proof is straightforward and shorter.

Lemma 2.3.3. Let R be a ring with unity 1 and $f : R \longrightarrow R$ is a function such that $f(1 + x) = f(x)$. If there exists an integer $m = m(x) \geq 1$ such that $x^m f(x) = 0$, then necessarily $f(x) = 0$.

Proof. For the elements x and $1 + x$, there exist integers $m = m(x) \geq 1$ and $n = n(1 + x) \geq 1$ such that

$$x^m f(x) = 0$$

and

$$(1 + x)^n f(1 + x) = 0 = (1 + x)^n f(x).$$

If $N = \max(m, n)$, then we have

$$x^N f(x) = 0. \tag{2.3.1}$$

$$(2.3.2) \qquad (1+x)^N f(1+x) = 0 = (1+x)^N f(x).$$

If $N = 1$, then the result follows trivially. Suppose $N \geq 2$. We have

$$\begin{aligned} f(x) &= [(1+x) - x]^{2N+1} f(x) \\ &= \{(1+x)^{2N+1} - {}^{2N+1}C_1 (1+x)^{2N} x + \ldots\ldots + (-1)^{2N+1} x^{2N+1}\} f(x) \\ &= 0, \text{ by (2.3.1) and (2.3.2).} \ \square \end{aligned}$$

Remark 2.3.4. Notice that commutator function $[x, y]$ satisfies the hypothesis of the lemma i.e., $[1 + x, y] = [x, y]$ and so the above lemma can be restated as follows :

Lemma 2.3.4. In a ring with unity 1, $x^m[x, y] = 0$ implies $[x, y] = 0$, for any positive integer $m = m(x, y) \geq 1$.

In preparation for the proof of our theorem, we first prove the follwing :

Lemma 2.3.5. Let R be a ring with unity 1 satisfying the identities (C_1) and (C_2). Then $U(R)$, the set of all invertible elements and $J(R)$, the Jacobson radical of R are commutative.

Proof. Since m and n are relatively prime, we may assume $rn - sm = 1$. for some positive integers r and s. If $k = sm$, then $k+1 = rn$ so that the identities (C_1) and (C_2) of the hypotheses imply that

$$(2.3.3) \qquad (xy)^k = (yx)^k, \text{ for all } x, y \in R.$$

and

(2.3.4) $$x^{k+1}y^{k+1} = y^{k+1}x^{k+1}, \text{ for all } x, y \in R.$$

Let $u, v \in U(R)$. Replacing x by u and y by $u^{-1}v$ in (2.3.3), we get

(2.3.5) $$uv^k = v^k u, \text{ for all } u, v \in R.$$

Now replacement of x by u and y by v in (2.3.4) yields $u^{k+1}v^{k+1} = v^{k+1}u^{k+1}$ and in view of (2.3.5), this implies that $uv = vu$, for all $u, v \in U(R)$. Hence $U(R)$ is commutative.

Further, let $a, b \in J(R)$. Then $1 + a$ and $1 + b$ are invertible and commute with each other. Thus $ab = ba$ and $J(R)$ is commutative. □

Lemma 2.3.6. Let R be a ring with unity 1 satisfying the identities (C_1) and (C_2). Then $R/J(R)$ is commutative.

Proof. $R/J(R)$ is semisimple. We know that every semisimple ring R is isomorphic to a subdirect sum of primitive rings R_α, each of which as a homomorphic image of R inherits the hypotheses placed on R and so we assume that $R/J(R)$ is primitive satisfying the hypotheses of Theorem 2.3.2. Notice that no complete matrix ring satisfies our hypotheses as consideration of $x = e_{1n}$ and $y = e_{n1}$ shows. Thus by the Jacobson Density Theorem [88, pp. 33], $R/J(R)$ is a division ring. Hence $R/J(R)$ is commutative by Lemma 2.3.5. □

Now we are ready to prove our theorem.

Proof of Theorem 2.3.2. By Lemma 2.3.6,

$$C(R) \subseteq J(R). \tag{2.3.6}$$

Replace x by u and y by $u^{-1}y$ in (2.3.3), to get $[u, y^n] = 0$, for all u in $U(R)$ and y in R. Now, if $a \in J(R)$, then $1 + a \in U(R)$. Replacing u by $1 + a$, we obtain

$$[a, y^n] = 0, \text{ for all } y \in R. \tag{2.3.7}$$

In view of (2.3.6), $[a, y^{n+1}] \in J(R)$ and hence commute with $u = 1 + a$, for $a \in J(R)$ by Lemma 2.3.5. Hence $0 = [u^{n+1}, y^{n+1}] = (n + 1)u^n[u, y^{n+1}]$, implies that $(n + 1)[u, y^{n+1}] = 0$. Replacing u by $1 + a$, we find

$$(n + 1)[a, y^{n+1}] = 0, \text{ for all } y \in R. \tag{2.3.8}$$

Using (2.3.7), we can assume $n[a, y^n] = 0$ and hence

$$n[a, y^n] = 0 = (n + 1)[a, y^{n+1}] = 0, \text{ for all } y \in R. \tag{2.3.9}$$

Since $J^2(R) \subseteq Z(R)$, the only terms in the expansion of $(y + a)^{n+1}$ which do not commute with y^{n+1} are those involving a exactly once. Hence

$$0 = [(y + a)^{n+1}, y^{n+1}]$$

$$0 = [y^n a + y^{n-1}ay + ... + yay^{n-1} + ay^n,\ y^{n+1}]$$

$$n(y^n a+y^{n-1}ay+.......+yay^{n-1}+ay^n)y^{n+1} = ny^{n+1}(y^n a+y^{n-1}ay+.......+yay^{n-1}+ay^n).$$

Using (2.3.8), we have

$$ny^{n+1}(y^n a+y^{n-1}ay+.......+yay^{n-1}+ay^n) = ny^{n+1}(y^n a+y^{n-1}ay+.......+ay^n).$$

$$ny^{n+1}(y^n a + y^{n-1}ay + ... + ay^n) = n(y^{2n}ay + ... + y^{n+1}ay) + nay^{2n+1}.$$

This gives $(ay^{2n+1} - y^{2n+1}a) = 0$ and hence by (2.3.9), $ny^{2n}[a, y] = 0$. By Lemma 2.3.4, this reduces to,

(2.3.10) $\qquad n[a, y] = 0,$ for all $y \in R$; $a \in J(R)$.

Replace y by y^n, to get $n[a, y^{n+1}] = 0$, which in view of (2.3.9) yields

(2.3.11) $\qquad [a, y^{n+1}] = 0,$ for all $y \in R$; $a \in J(R)$.

By using (2.3.7) and (2.3.11), we have $[a, y]y^n = 0$. Application of Lemma 2.3.4 yields that $[a, y] = 0$, for all $y \in R$ and $a \in J(R)$ i.e., $J(R) \subseteq Z(R)$. Consequently, (2.3.6) gives

(2.3.12) $\qquad C(R) \subseteq J(R) \subseteq Z(R).$

The identity (C_1) of the hypothesis implies that $x(xy)^n = x(yx)^n = (xy)^n x$ i.e., $[x, (xy)^n] = 0$. By Lemma 2.3.6 and (2.3.12), we have $(xy)^n - x^n y^n \in J(R) \subseteq Z(R)$ and hence $[x, (xy)^n] = x^n[x, y^n] = 0$. Thus by Lemma 2.3.4, $[x, y^n] = 0$. for all $x, y \in R$. Application of Lemma 2.3.2 yields that $0 = [x, y^n] = ny^{n-1}[x, y]$. Lemma 2.3.4 implies that

(2.3.13) $\qquad n[x, y] = 0,$ for all $x, y \in R$.

In view of (2.3.4), $[x^{n+1}, y^{n+1}] = 0$, which together with (2.3.13) and Lemma 2.3.2 imply that $(n + 1)x^n[x, y^{n+1}] = 0$, for all $x, y \in R$. By

Lemma 2.3.4, $(n+1)[x, y^{n+1}] = 0$, for all $x, y \in R$. Arguing in the same fashion again, we have

$$(n+1)^2[x, y] = 0, \text{ for all } x, y \in R,$$

i.e.,

$$n^2[x, y] + 2n[x, y] + [x, y] = 0, \text{ for all } x, y \in R.$$

Hence $[x, y] = 0$, for all $x, y \in R$ by (2.3.13). This completes the proof of the theorem. □

Remark 2.3.5. The following example demonstrates that if m and n are not relatively prime in the hypotheses of the above theorem, the ring may be badly noncommutative.

Example 2.3.2. The ring R given in Example 2.2.1, contains unity and satisfies the conditions $[x^2, y^2] = 0$ and $(xy)^2 = (yx)^2$, for all $x, y \in R$ but $(xy)^3 \neq (yx)^3$. However, R is a noncommutative ring.

Remark 2.3.6. The following example justifies that the conditions imposed on the hypotheses of Theorem 2.3.2 are not superfluous.

Example 2.3.3. Let $R = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \mid a, b, c, d \in GF(3) \right\}$. It can be easily verified that R is a noncommutative ring satisfying the condition $[x^3, y^3] = 0$, for all $x, y \in R$. However, R does not satisfy the condition $(xy)^2 = (yx)^2$, for all $x, y \in R$.

2.4 A GENERALIZATION OF JACOBSON THEOREM

In an attempt to generalize the classical theorem of Wedderburn [124] that a finite division ring must be commutative and also the result that a Boolean ring (satisfying $x^2 = x$) is necessarily commutative, Jacobson [88] established in 1945 :

Theorem J. Let R be a ring in which for every $x \in R$, there exists an integer $n = n(x) > 1$ such that $x^n = x$. Then R is commutative.

The above result because of its intrinsic beauty and wide applications sparked off tremendous interest among researchers which has been still continuing. I. N. Herstein generalized Theorem J stage by stage in a number of papers (cf. [66], [67], [69]) and finally proved the following : If corresponding to every element x of a ring R, there exists a polynomial $f(x)$ with integral coefficients such that $x^2 f(x) - x$ is central, then the ring R must be commutative.

Let $\mathbb{Z}[X]$ denote the ring of all polynomials in the indeterminate X over the ring $\mathbb{Z}$ of integers and $f(x)$ denote the ring element in R which is obtained upon substituting x for X in polynomial $f(X) \in \mathbb{Z}[X]$. Then the mentioned result can be restated as follows :

Theorem H^*. If for each $x, y \in R$, there exists a polynomial $f(X) \in X^2\mathbb{Z}[X]$ such that $[x - f(x), y] = 0$, then R is commutative.

One of the natural extensions of Theorem H^* and in turn Theorem J can be obtained by considering the related property :

$(*)$ For each $x, y \in R$, there exist polynomials $f(X),\ g(X),\ h(X) \in X^2\mathbb{Z}[X]$ such that $(1 - g(x^m y))[x^m y - x^n f(x^m y)x^j,\ x](1 - h(x^m y)) = 0$, where $m \geq 0,\ n \geq 0,\ j \geq 0$ are non-negative integers.

Before investigating the commutativity of rings with the above property we pause to give a classification of all noncommutative rings due to W. Streb [118, corollary 1].

Lemma 2.4.7. Let $(*)$ be a ring property which is inherited by factor subrings. Then every ring with unity 1 satisfying $(*)$ is commutative, if no rings either of the following types satisfy $(*)$:

(a) $\begin{pmatrix} GF(p) \ , & GF(p) \\ 0 \ , & GF(p) \end{pmatrix}$, p a prime.

(b) $\begin{pmatrix} GF(p) \ , & GF(p) \\ 0 \ ; & 0 \end{pmatrix}$, p a prime.

(c) $\begin{pmatrix} GF(p) \ ; & 0 \\ GF(p) \ , & 0 \end{pmatrix}$, p a prime.

(d) $M_\sigma(K)=\left\{\begin{pmatrix} \alpha & \beta \\ 0 & \sigma(\alpha) \end{pmatrix} \mid \alpha, \beta \in K)\right\}$, where K is a finite field with a nontrivial automorphism σ.

(e) A noncommutative division ring.

(f) $S = <1> + T$, T a noncommutative radical subring of S.

(g) $S = <1> + T$, T a noncommutative subring of S such that $T[T, T] = [T, T] = 0$.

The proof of the following lemma can be found in [91, Lemma 1].

Lemma 2.4.8. R be a ring with unity 1 in which for every $x, y \in R$, there exist polynomials $f(X), g(X) \in X^2\mathbb{Z}[X]$ such that $[x - f(x), y - g(y)] = 0$. If R

is noncommutative, then there exists a factor subring of R which is of type (a) or (b).

Lemma 2.4.9. R be a division ring satisfying $(*)$. Then R is commutative.

Proof. Let u be a unit in R. Then for each $y \in R$, there exist polynomials $f(X), g(X), h(X) \in X^2\mathbb{Z}[X]$ such that

$$0 = (1 - g(u^m u^{-m} y))[u^m u^{-m} y - u^n f(u^m u^{-m} y) u^j, u](1 - h(u^m u^{-m} y)).$$

$$= (1 - g(y))[y - u^n f(y) u^j, u](1 - h(y)).$$

This implies that either $y - yg(y) = 0$, $y - yh(y) = 0$ or $[y - u^n f(y)u^j, u] = 0$, for some $f(X), g(X), h(X) \in X^2\mathbb{Z}[X]$. In the first two cases R is commutative by Theorem H^*. Henceforth, we shall assume that $[y - u^n f(y)u^j, u] = 0$ that is,

$$(2.4.1) \qquad [u, y] = u^n [u, f(y)] u^j.$$

Further, choose a polynomial $f(X) \in X^2\mathbb{Z}[X]$ such that $[u^{-1}, y] = u^{-n}[u^{-1}, f(y)]u^{-j}$. This implies that

$$(2.4.2) \qquad u^n [u, y] u^j = [u, f_1(y)].$$

By (2.4.2), there exists a polynomial $f_2(X) \in X^2\mathbb{Z}[X]$ such that $[u, f_1(y)] = u^n[u, f_2(f_1(y))]u^j$. Thus in view of (2.4.2), this yields that $u^n[u, f_3(y)]u^j = u^n[u, y]u^j$, where $f_3(X) = f_2(f_1(X)) \in X^2\mathbb{Z}[X]$. This yields that $[u, y - f_3(y)] = 0$, for some $f_3(X) \in X^2\mathbb{Z}[X]$ and by Theorem H^*, R is commutative. $\square$

We are now well equipped to prove the following theorem which is a wide generalization of Theorems J, H^* and many others.

Theorem 2.4.3. Let R be a ring with unity 1 satisfying the property $(*)$. Then R must be commutative.

Proof. By e_{ij} , we represent as usual the elementary matrix with 1 at the intersection of i^{th} row and j^{th} column and 0 elsewhere. Suppose that R satisfies $(*)$.

First we consider the ring of type (a), then we see that in $(GF(p))_2$, p a prime for each $f(X), g(X), h(X) \in X^2\mathbb{Z}[X]$,

$$(1 - g(e_{11}^m e_{12}))[e_{11}^m e_{12} - e_{11}^n f(e_{11}^m e_{12})e_{11}^j, e_{11}](1 - h(e_{11}^m e_{12})) = -e_{12} \neq 0,$$

a contradiction. Hence no rings of type (a) satisfy $(*)$.

Now consider the ring $M_\sigma(K)$, a ring of type (b) and choose $x = \begin{pmatrix} \alpha & 0 \\ 0 & \sigma(x) \end{pmatrix}$, $y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $(\alpha \neq \sigma(x))$. Then for each $f(X), g(X), h(X) \in X^2\mathbb{Z}[X]$, we see that

$$(1 - g(x^m y))[x^m y - x^n f(y)x^j, x](1 - h(x^m y)) = \alpha^m(\sigma(\alpha) - \alpha)e_{12} \neq 0.$$

Thus no rings of type (b) satisfy $(*)$.

Next, if R is a ring of type (c), then by Theorem H^*, R is commutative, a contradiction.

Now, let R be a ring of type (d). A careful scrutiny of the proof of Lemma 2.4.9 shows that for an unit u and arbitrary $y \in R$, there exist polynomials $f_3(X), g(X), h(X) \in X^2\mathbb{Z}[X]$ such that either $y - g(y) = 0$, $y - h(y) = 0$ or $[y - f_3(y), y] = 0$. Now, let $a, b \in T$. Then $1 + a$ is a unit and there exist polynomials $f_3(X), g(X), h(X) \in X^2\mathbb{Z}[X]$ such that either $b - bg(b) = 0$, $b - bh(b) = 0$ or $[b - f_3(b), 1 + a] = 0$ and in all cases T is commutative. Thus R can not be of type (d).

Finally, suppose that R is a ring of type (e). Then for each $a, b \in T$, there exist polynomials $f(X), g(X), h(X) \in X^2\mathbb{Z}[X]$ such that

$$
\begin{aligned}
0 &= (1-g((1+a)^m b))[(1+a)^m b-(1+a)^n f((1+a)^m b)(1+a)^j,\ 1+a](1-h((1+a)^m b)) \\
&= (1 - g((1 + a)^m b))[(1 + a)^m b,\ 1 + a](1 - h((1 + a)^m b)) \\
&= (1 - g((1 + a)^m b))[b,\ 1 + a](1 - h((1 + a)^m b)) \\
&= [a, b].
\end{aligned}
$$

This is a contradiction and hence no rings of type (e) satisfy $(*)$.

Thus, in view of Lemma 2.4.7, R must be commutative. □

If the integral exponents m, n, j in the property $(*)$ are allowed to vary with the pair of elements $x, y \in R$, then a closer look at the proof of the above theorem shows that R has no factor subring of type (a) or (b) and we can not conclude the result of Theorem 2.4.3. However, in view of Lemma 2.4.8, we have the following :

Theorem 2.4.4. Let R be a ring with unity 1 satisfying the property

$(**)$ For each pair of elements $x, y \in R$, there exist inegers $m = m(x, y) \geq 0$, $n = n(x, y) \geq 0$, $j = j(x, y) \geq 0$ and polynomials $f(X)$, $g(X)$, $h(X)$ in $X^2\mathbb{Z}[X]$ such that $(1 - g(x^m y))[x^m y - x^n f(x^m y)x^j,\ x](1 - h(x^m y)) = 0$.

If in addition, $[x - f(x), y - g(y)] = 0$, for all $x, y \in R$, then R must be commutative.

Remark 2.4.7. We could not succeed to replace the additional condition $[x - f(x), y - g(y)] = 0$ in Theorem 2.4.4 by some other simpler condition. Nevertheless, we conjecture that some torsion condition or absence of nonzero nilpotent (nil) commutator ideal may buy the commutativity of rings with $(**)$.

Remark 2.4.8. The following example is sufficient to show that the restriction of the existence of unity in the rings of Theorem 2.2.1, 2.3.2 and 2.4.4 is not superfluous.

Example 2.4.4. Let $R=\left\{\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} \mid a, b, c \in GF(3)\right\}$. Then R is a 2-torsion free ring without unity satisfying the conditions of the mentioned theorems. However, R is not commutative.

Chapter-3

STRUCTURE OF CERTAIN RINGS

3.1 INTRODUCTION

The Boolean condition ($x^2 = x$, for all ring elements x) has been extended in various directions during the second half of the 20^{th} century and rings with these generalized conditions have been studied by many algebraists like Jacobson, Herstein, Ligh, Luh, Bell, Abujabal, Tominaga, Yaqub, Quadri and Ashraf ; to mention a few. In 1986, Sercoid and MacHale [117] established that a ring R satisfying the polynomial identity $(xy)^2 = xy$, for all $x, y \in R$, is necessarily commutative. Later, Ligh and Luh [100] obtained a direct sum decomposition of such rings, which was subsequently sharpened by Bell and Ligh [38]. These results are jumping-off point for the work presented in this chapter.

In section 3.2, we consider more general polynomial conditions and establish a direct sum decomposition of rings under these conditions. In the next section, we extend the result of section 3.2 by taking slightly more restricted polynomial constraints.

3.2 DECOMPOSITION OF CERTAIN PERIODIC RINGS

One of the natural extensions of Boolean condition is $x^n = x$, for some positive integer $n = n(x) > 1$. In 1945, Jacobson [87] established that a ring with the mentioned generalized Boolean condition turns out to be commutative. We shall call such rings as *J-rings*. The notion of J-rings is further extended as periodic rings :

Definition 3.2.1 (Periodic Ring). A ring R is called periodic if for every element $x \in R$, there exist distinct positive integers $m = m(x)$ and $n = n(x)$ such that $x^m = x^n$.

A sufficient condition for a ring R to be periodic is Chacron's criterion [52]: For each $x \in R$, there exist an integer $m = m(x) \geq 1$ and a polynomial $f(X) \in \mathbb{Z}[X]$, the ring of polynomials in X over the ring $\mathbb{Z}$ of integers such that $x^m = x^{m+1}f(x)$.

In view of the concept of a J-ring, the notion of idempotent elements can also be naturally extended as follows :

Definition 3.2.2 (Potent Element). An element x of a ring R is said to be potent if there exists a positive integer $n = n(x) > 1$ such that $x^n = x$.

Thus J-rings are those whose every element is potent.

Another generalization of Boolean condition may be $(xy)^2 = xy$ which can be further weakend as $(xy)^n = xy$, for all $x, y \in R$ where $n = n(x, y)$ is a positive integer. The example of zero ring i.e., the ring with trivial multiplication ($xy = 0$) demonstrates that there exist non J-rings satisfying $(xy)^n = xy$ with $n = n(x, y) > 1$. In 1986, Searcoid and MacHale [117] established commutativity of rings satisfying the mentioned condition. Further, Ligh and Luh [100] obtained a direct sum decomposition of these rings. Recently, Bell and Ligh [38] sharpened the result and obtained a decomposition for rings with the property $xy = (xy)^2p(x, y)$ or

$xy = (yx)^2 p(y,x)$, where $p(X,Y) \in \mathbb{Z}[X,Y]$, the ring of polynomials in two noncommuting indeterminates over the ring $\mathbb{Z}$ of integers. In fact, the authors proved the following :

Theorem BL. Suppose that for each $x, y \in R$, there exists a polynomial $p(X,Y) \in \mathbb{Z}[X,Y]$ such that $xy = (xy)^2 p(x,y)$. Then R is a direct sum of a J-ring and a zero ring.

In this section, we obtain the structure of rings satisfying either of the following related ring properties :

(P_1) For every pair of elements x, y in R, there exist an integer $n = n(x,y) > 1$ and $p(X,Y) \in \mathbb{Z}[X,Y]$ such that $xy = (xy)^n p(x,y)$.

(P_2) For every pair of elements x, y in R, there exist an integer $n = n(x,y) > 1$ and $p(X,Y) \in \mathbb{Z}[X,Y]$ such that $xy = (yx)^n p(x,y)$.

(P_3) For every pair of elements x, y in R, there exist integers $m = m(x,y) > 1$; $n = n(x,y) > 1$ and $p(X,Y) \in \mathbb{Z}[X,Y]$ such that $xy = x^m y^n p(x,y)$.

(P_4) For every pair of elements x, y in R, there exist integers $m = m(x,y) > 1$; $n = n(x,y) > 1$ and $p(X,Y) \in \mathbb{Z}[X,Y]$ such that $xy = y^m x^n p(x,y)$.

(P_5) For every pair of elements x, y in R, there exist integers $m = m(x,y) > 1$; $n = n(x,y) > 1$ and $p(X,Y) \in \mathbb{Z}[X,Y]$ such that $xy = x^m p(x,y) x^n$.

(P_6) For every pair of elements x, y in R, there exist integers $m = m(x,y) > 1$; $n = n(x,y) > 1$ and $p(X,Y) \in \mathbb{Z}[X,Y]$ such that

$$xy = y^m p(x, y) y^n.$$

It can be easily observed that a ring satisfying any one of the properties $(P_1) - (P_6)$ is necessarily periodic. Let us denote the set of all potent elements of a ring R by P and the set of all nilpotent elements of R by N.

Theorem 3.2.1. Let R be a ring satisfying either of the conditions $(P_1) - (P_6)$. Then $R = P \oplus N$.

In order to develop the proof of our theorem, we begin with the following lemmas which can be found in [28] and [33] respectively.

Lemma 3.2.1. Let R be a periodic ring. Then each element $x \in R$ can be represented as sum of a potent element and a nilpotent element.

Lemma 3.2.2. If in a periodic ring R, every element $x \in R$ can be expressed uniquely in the form $x = a + u$, where $a \in P$ and $u \in N$, then P and N both are ideals and $R = P \oplus N$.

Now we prove the following :

Lemma 3.2.3. Let R be a ring satisfying either of the conditions $(P_1) - (P_6)$. Then N is an ideal of R.

Proof. Let R satisfy (P_1). By taking $y = x$ in condition (P_1), we get $h(x) \in x^2\mathbb{Z}[x]$ such that

$$x^2 = x^{2k} h(x); \text{ for } k = k(x) \geq 2. \tag{3.2.1}$$

Let $u \in N$. Then by (3.2.1), we have

$$u^2 = u^{2m}q(u) = u^{2m}u^2 q_1(u)$$

$$= (u^{2m})^2 q(u)q_1(u) = (u^{2m})^2 u^2 q_2(u)q_1(u)$$

$$= (u^{2m})^3 q(u)q_2(u)q_1(u)$$

....

$$= (u^{2m})^t u^2 q_t(u)q_{t-1}(u)q_{t-2}(u).......q_1(u)$$

$$u^2 = 0, \quad \text{for sufficiently large t.}$$

Let $u, v \in N$. Then we have $(u - v)^2 = u^2 - uv - vu + v^2$. Since $u^2 = 0$, for every $u \in N$, $uv = -vu$, for every $u, v \in N$. This implies that $(u - v)^3 = 0$, for all $u, v \in N$. Hence $u - v \in N$.

It is easy to see that R satisfying (P_1) is zero-commutative. Indeed, if $xy = 0$, then

$$yx = (yx)^{n'}p(y, x) = 0, \quad n' = n'(y, x)$$

$$= (yxyxyx..........yx)p(y, x)$$

$$= y(xy)(xy)...........(xy)xp(y, x)$$

$$= 0.$$

Now using the fact that $u^2 = 0$, for all $u \in N$ and R is zero-commutative, we find that $uv \in N$, for all $u, v \in N$. Hence N is an ideal of R. □

Similarly we can prove for the other cases.

Proof of Theorem 3.2.1. First, suppose that R satisfies (P_1). Let $u \in N$ and $x \in R$. Then by condition (P_1), we have $xu = (xu)^l p(x, u) = 0$, for all $u \in U$,

$x \in R$ and zero-commutativity yields that $ux = 0$, for all $u \in N$, $x \in R$. Hence

$$(3.2.2) \qquad RN = NR = \{0\}.$$

Since R is periodic, every element $x \in R$ can be written as $x = a + u$, where $a \in P$ and $u \in N$ by Lemma 3.2.1. To complete the proof, it is sufficient to show that the above representation is unique. Let $a + u = b + v$, for some $a, b \in P$ and $u, v \in N$. Then

$$(3.2.3) \qquad a - b = v - u.$$

Since $a, b \in P$, there exist integers $r = r(a) > 1$ and $s = s(b) > 1$ such that $a^r = a$ and $b^s = b$. Take $k = (r-1)s - (r-2) = (s-1)r - (s-2)$. Then it is clear that $a^k = a$ and $b^k = b$. Note that $e = a^{k-1}$ and $f = b^{k-1}$ are idempotents with $ea = a$ and $fb = b$. Multiplying (3.2.3) by a and b on both sides and using the relation (3.2.2), we get $a^2 = ab = ba$ and $b^2 = ab = ba$, which yield that $a^2 = b^2$ and $e = f$. Again multiply (3.2.3) by e, to get $a = b$. Hence the theorem is proved. □

Similarly we can prove for the other cases.

3.3 DECOMPOSITION OF RINGS UNDER CERTAIN CONSTRAINTS INVOLVING A SUBSET

Now the question arises what we can say about the direct sum decomposition of the ring R if the underlying conditions are assumed to be satisfied by certain restricted elements of R. In this direction, we consider the following conditions. Suppose A is a subset of R and $R \backslash A = \{x \in R \mid x \notin A\}$.

$(P_1)^*$ For every pair of elements x, y in $R\backslash A$, there exist an integer $n = n(x,y) > 1$ and $p(X,Y) \in \mathbb{Z}[X,Y]$ such that $xy = (xy)^n p(x,y)$.

$(P_2)^*$ For every pair of elements x, y in $R\backslash A$, there exist an integer $n = n(x,y) > 1$ and $p(X,Y) \in \mathbb{Z}[X,Y]$ such that $xy = (yx)^n p(x,y)$.

$(P_3)^*$ For every pair of elements x, y in $R\backslash A$, there exist integers $m = m(x,y) > 1$; $n = n(x,y) > 1$ and $p(X,Y) \in \mathbb{Z}[X,Y]$ such that $xy = x^m y^n p(x,y)$.

$(P_4)^*$ For every pair of elements x, y in $R\backslash A$, there exist integers $m = m(x,y) > 1$; $n = n(x,y) > 1$ and $p(X,Y) \in \mathbb{Z}[X,Y]$ such that $xy = y^m x^n p(x,y)$.

$(P_5)^*$ For every pair of elements x, y in $R\backslash A$, there exist integers $m = m(x,y) > 1$; $n = n(x,y) > 1$ and $p(X,Y) \in \mathbb{Z}[X,Y]$ such that $xy = x^m p(x,y) x^n$.

$(P_6)^*$ For every pair of elements x, y in $R\backslash A$, there exist integers $m = m(x,y) > 1$; $n = n(x,y) > 1$ and $p(X,Y) \in \mathbb{Z}[X,Y]$ such that $xy = y^m p(x,y) y^n$.

Theorem 3.3.2. Let R be a ring with $N \neq \{0\}$ and A be an additive subgroup of R with $A \subseteq N$. If for each $x, y \in R\backslash A$ either of the conditions $(P_1)^* - (P_6)^*$ holds, then $R = P \oplus N$.

To develop the proof of the above theorem, we prove the following lemma:

Lemma 3.3.4. Let R be a ring satisfying either of the conditions $(P_1)^* - (P_6)^*$, where A is the additive subgroup of R with $A \subseteq N$. Then $RN = NR = \{0\}$.

Proof. We look at each of the conditions in turn :

(i) Let R satisfy $(P_1)^*$. Then using the similar arguments as we have done in the proof of Theorem 3.2.1, we get

$$x^2 = 0, \quad \text{for } x \in N\backslash A. \tag{3.3.1}$$

We show that if $x \in N\backslash A$, $y \in R$ and $xy = 0$, then $yx = 0$.
Suppose $x \in N\backslash A$, $y \in R\backslash A$ and $xy = 0$. Then it can be easily obtained by using condition $(P_1)^*$. Further, assume that $xz = 0$, for all $x \in N\backslash A$ and $z \in A$. Then $x + z \notin A$ and $x(x + z) = x^2 + xz = 0$. Thus $(x + z)x = 0 = x^2 + zx = zx$.

For any $a, b \in A$, we see that $a - b \in A \subseteq N$. Let $x \in N\backslash A$, so that $x^2r = 0$, for all $x \in N\backslash A$ and $r \in R$. Hence we get

$$xRx = \{0\}. \tag{3.3.2}$$

Suppose $y \in N$. Then $y^s = 0$, for some integer $s > 1$ and we have $(x - y)^{2s} = 0$, for all $x, y \in N$. This implies that $x - y \in N$. Let $x \in N\backslash A$. Then by (3.3.2), $(xr)^2 = 0$, for all $x \in N\backslash A$ and $r \in R$ i.e.,

$$xr \in N\backslash A \text{ for all } x \in N\backslash A \text{ and } r \in R. \tag{3.3.3}$$

Assume that $x \in N\backslash A$ and $z \in A$. Since $x + z \in N$ and $xr \in N\backslash A$, we can write $zr = (x + z)r - xr \in A$ which yields that

$$zr \in A, \quad \text{for all } z \in A \text{ and } r \in R. \tag{3.3.4}$$

From (3.3.3) and (3.3.4), we obtain $yr \in N$, for all $y \in N$ and $r \in R$. Similarly, we can show that $ry \in N$, for all $y \in N$ and $r \in R$. Hence N is an ideal.

Now we show that N annihilates R on both sides. Let $x \in R\backslash A$ and $u \in N\backslash A$. Then there exists an integer $m = m(x,u) > 1$ such that $xu = (xu)^m q(xu)$. Thus by (3.3.1), we find that $xu = 0$, for $x \in R\backslash A$ and $u \in N\backslash A$. Hence by zero-commutativity, we obtain

$$(3.3.5) \qquad R\backslash A \; N\backslash A = N\backslash A \; R\backslash A = \{0\}.$$

Since every element of A is a difference of two elements of $R\backslash A$ i.e., $x - y \in A$, for $x, y \in R\backslash A$. We have $(x-y)u = xu - yu = 0$, for all $x, y \in R\backslash A$ and $u \in N\backslash A$. Using (3.3.5), we get $u(x-y) = 0$, for all $x, y \in R\backslash A$ and $u \in N\backslash A$. Hence

$$(3.3.6) \qquad A \; N\backslash A = N\backslash A \; A = \{0\}.$$

From (3.3.5) and (3.3.6), we obtain

$$(3.3.7) \qquad R \; N\backslash A = N\backslash A \; R = \{0\}.$$

Since every element of A is also a difference of two elements of $N\backslash A$ i.e., $x - y \in A$, for $x, y \in N\backslash A$, we have $(x-y)r = xr - yr = 0$, for all $x, y \in N\backslash A$ and $r \in R$. Using (3.3.7), we get $r(x-y) = 0$, for all $x, y \in N\backslash A$ and $r \in R$. Thus we find that

$$(3.3.8) \qquad RA = AR = \{0\}.$$

Combining the relations (3.3.7) and (3.3.8), we get

$$RN = NR = \{0\}.$$

In case R satisfies $(P_2)^*$ argue as above.

(ii) Let R satisfy $(P_3)^*$. Replacing y by x in $(P_3)^*$, we get $f(x) \in X^2\mathbb{Z}[X]$ such that

$$x^2 = x^{m+n}f(x), \quad \text{for } m = m(x) > 1 \; ; \; n = n(x) > 1. \tag{3.3.9}$$

Let $x \in N\backslash A$. Then by (3.3.9), we obtain

$$\begin{aligned} x^2 &= x^{m+n}q(x) = x^{m+n}x^2q_1(x) \\ &= (x^{m+n})^2q(x)q_1(x) = (x^{m+n})^2x^2q_2(x)q_1(x) \\ &= (x^{m+n})^3q(x)q_2(x)q_1(x) = (x^{m+n})^3x^2q_3(x)q_2(x)q_1(x) \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ x^2 &= (x^{m+n})^tx^2q_t(x)q_{t-1}(x).....q_1(x) \\ x^2 &= 0, \quad \text{for sufficiently large t.} \end{aligned}$$

Let $u \in N\backslash A$ and $x \in R\backslash A$. Then by condition $(P_3)^*$, we have $xu = x^mu^np'(x,u) = 0$, for all $u \in N\backslash A$ and $x \in R\backslash A$. Thus zero-commutativity yields that $ux = 0$, for all $u \in N\backslash A$ and $x \in R\backslash A$. Hence $R\backslash A\ N\backslash A = N\backslash A\ R\backslash A = \{0\}$. Further, arguing in the similar manner as we have done in case (i), we get the required result.

In case R satisfies $(P_4)^*$ argue as above.

(iii) Let R satisfy $(P_5)^*$. Replacing y by x in $(P_5)^*$, we get $g(X) \in X^2\mathbb{Z}[X]$ such that

$$(3.3.10) \qquad x^2 = x^m g(x) x^n, \quad \text{for } m = m(x) > 1 \ ; \ n = n(x) > 1.$$

Let $x \in N \backslash A$. Then by (3.3.10), we obtain

$$\begin{aligned} x^2 &= x^m q(x) x^n = x^m x^2 q_1(x) x^n \\ &= x^{2m} q(x^2) x^n q_1(x) x^n = x^{2m} x^2 q_2(x) x^n q_1(x) x^n \\ &= x^{3m} q(x^2) x^n q_2(x) x^n q_1(x) x^n = x^{3m} x^2 q_3(x) x^n q_2(x) x^n q_1(x) x^n \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ x^2 &= x^{tm} x^2 q_t(x) x^n q_{t-1}(x) x^n \ldots\ldots q_1(x) x^n \\ x^2 &= 0, \quad \text{for sufficiently large t.} \end{aligned}$$

Using the similar arguments as we have done in previous case, we find that $RN = NR = \{0\}$. □

In case R satisfies $(P_4)^*$ argue as above.

Now we are ready to prove our theorem.

Proof of Theorem 3.3.2. Let R satisfy $(P_1)^*$. Replacing y by x in $(P_1)^*$, we get $p'(x) \in X^2\mathbb{Z}[X]$ such that

$$x^2 \ = \ x^{2n} p'(x), \text{ for all } x \in R \backslash A \text{ and } n = n(x) \ > \ 1.$$

Since every element of A is nilpotent, for each $x \in A$, $x^{n'} = 0$ for some integer $n' > 1$ and we can write $x^{n'} = x^{n'+m'} = 0$ for some $m' > 1$. Hence R is periodic by Chacron's criterion. Similarly R is periodic in the other cases. Now using the same arguments as in case of Theorem 3.2.1, we get the required result. □

In view of Lemma 3.3.4, we conclude that the nilpotent elements of R annihilate R on both sides and hence central. Since J-rings are commutative (cf. Theorem J), the consequence of the above theorem leads to the following corollary which generalizes the result due to Tominaga and Yaqub [119, Theorem 2].

Corollary 3.3.1. Let R be a ring with $N \neq \{0\}$ and A be an additive subgroup of R with $A \subseteq N$. If for each $x, y \in R \backslash A$ either of the conditions $(P_1)^* - (P_6)^*$ holds, then R is commutative.

Chapter-4

STRUCTURE AND COMMUTATIVITY OF CERTAIN NEAR RINGS

4.1 INTRODUCTION

Throughout this chapter, by a near ring we shall mean a left near ring i.e., a near ring satisfying necessarily left distributive law $a(b+c) = ab + ac$, for all elements a, b and c. As in case of rings, $Z(R)$ denotes the multiplicative centre of near ring R.

It is not easy to obtain near ring theoretic analogous of ring theoretic results. Neither do many of them hold in general. For instance, there exist Boolean near rings ($x^2 = x$) which are not commutative (cf. Example 1.3.1 (ii)). However, some of the results have been proved which are generalizations of some of the well-known commutativity theorems in rings to near rings ; to mention a few ; [6], [23], [31], [111] and [112]. As pointed out in previous chapter, Bell and Ligh [38] obtained a decomposition theorem for rings with the property $xy = (xy)^2 f(x,y)$, where $f(X,Y) \in \mathbb{Z}(X,Y)$, the ring of polynomials in two noncommuting indeterminates over the ring $\mathbb{Z}$ of integers. In the same paper by remarking that in case of near rings the analogous hypotheses do not quite yield the direct sum decomposition (c.f. Example 4.2.1), the authors have defined a weaker notion of orthogonal sum : *A near ring R is an orthogonal sum of subnear rings A and B if $AB = BA = \{0\}$ and each element of R has a unique representation in the form $a + b$ with $a \in A$ and $b \in B$. Such a sum may be denoted by $R = A \uplus B$.*

In section 4.2, we obtain an orthogonal sum decomposition for a $d-g$ near ring R satisfying any one of the following conditions: (I) $xy = (xy)^n p(xy)$, (II) $xy = x^m y^n p(xy)$, (III) $xy = y^m x^n p(xy)$, (IV) $xy = x^m p(xy) x^n$, (V) $xy = y^m p(xy) y^n$, (I^*) $xy = (xy)^n p(yx)$, (II^*) $xy = x^m y^n p(yx)$, (III^*) $xy = y^m x^n p(yx)$, (IV^*) $xy = x^m p(yx) x^n$ and (V^*) $xy = y^m p(yx) y^n$, where $p(xy)$ denotes an element of R which is a finite sum of powers $(xy)^k$, for $k \geq 2$ and additive inverses of such powers.

In 1976, Ligh and Luh [99] introduced the concept of D-near rings and pointed out that there exist non $d-g$ near rings which are D-near rings (c.f. Example 4.3.2). In section 4.3, we attempt to extend the result of section 4.2 to that wider class of near rings.

Part of recent work on near rings has been concerned with sufficient conditions for near rings to be rings. Ligh [97], by proving that distributively generated Boolean near rings are rings has introduced the possibility that some of the more complicated polynomial identity conditions implying commutativity in rings may turn distributively generated near rings into rings. However, Example 4.2.1 rules out this possibility also in general. In the last section, we continue the study considering $d-g$ near ring satisfying conditions (I) $-$ (V^*).

4.2 STRUCTURE OF $d-g$ NEAR RINGS

In the previous chapter, we have obtained a direct sum decomposition for the rings satisfying either of the conditions (P_1) $-$ (P_6).

One may ask whether the analogous hypotheses yield a direct sum decomposition in the case of near rings. The following example due to

Clay (cf. [54, Example 2.5 # 29]) shows that it is not possible to obtain such a decomposition for near rings in general.

Example 4.2.1. Let $R = \{0,\ a,\ b,\ c,\ u,\ v\}$. In R, define addition and multiplication as follows :

+	0	a	b	c	u	v
0	0	a	b	c	u	v
a	a	0	v	u	c	b
b	b	u	0	v	a	c
c	c	v	u	0	b	a
u	u	b	c	a	v	0
v	v	c	a	b	0	u

.	0	a	b	c	u	v
0	0	0	0	0	0	0
a	0	a	a	a	0	0
b	0	a	a	a	0	0
c	0	a	a	a	0	0
u	0	0	0	0	0	0
v	0	0	0	0	0	0

It can be readily verified that $(R, +)$ is a non-abelian group and $(R, +, \cdot)$ is a near ring satisfying $xy = xy^2x$, for all $x, y \in R$, a particular form of the condition (P_5) given in Chapter 3. However, the totality $P = \{0, a\}$ of potent elements of R is not a normal subgroup of $(R, +)$ and hence fails to be an ideal of R.

Thus for near rings, the analogous hypotheses do not quite yield a direct sum decomposition. Despite this bad behaviour of near rings we should not give up our attempt to investigate structure of near rings under similar polynomial conditions (may be slightly different). For this purpose, we define a weaker notion of orthogonal sum as follows :

Definition 4.2.1 (Orthogonal Sum). A near ring R is an orthogonal sum of subnear rings A and B if $AB = BA = \{0\}$ and each element of R has a unique representation in the form $a + b$ with $a \in A$ and $b \in B$. Such a sum may be denoted by $R = A \uplus B$.

In this section, we shall consider the following conditions, where $p(xy)$ denotes an element of R which is a finite sum of powers $(xy)^k$ and additive inverses of such powers for $k \geq 2$.

(I) For every pair of elements x, y in R, there exists a positive integer $n = n(x, y) \geq 1$ such that $xy = (xy)^n p(xy)$.

(II) For every pair of elements x, y in R, there exist positive integers $m = m(x, y) \geq 1$ and $n = n(x, y) \geq 1$ such that $xy = x^m y^n p(xy)$.

(III) For every pair of elements x, y in R, there exist positive integers $m = m(x, y) \geq 1$ and $n = n(x, y) \geq 1$ such that $xy = y^m x^n p(xy)$.

(IV) For every pair of elements x, y in R, there exist positive integers $m = m(x, y) \geq 1$ and $n = n(x, y) \geq 1$ such that $xy = x^m p(xy) x^n$.

(V) For every pair of elements x, y in R, there exist positive integers $m = m(x, y) \geq 1$ and $n = n(x, y) \geq 1$ such that $xy = y^m p(xy) y^n$.

(I^*) For every pair of elements x, y in R, there exists a positive integer $n = n(x, y) \geq 1$ such that $xy = (xy)^n p(yx)$.

(II^*) For every pair of elements x, y in R, there exist positive integers $m = m(x, y) \geq 1$ and $n = n(x, y) \geq 1$ such that $xy = x^m y^n p(yx)$.

(III^*) For every pair of elements x, y in R, there exist positive integers $m = m(x, y) \geq 1$ and $n = n(x, y) \geq 1$ such that $xy = y^m x^n p(yx)$.

(IV^*) For every pair of elements x, y in R, there exist positive integers $m = m(x, y) \geq 1$ and $n = n(x, y) \geq 1$ such that $xy = x^m p(yx) x^n$.

(V^*) For every pair of elements x, y in R, there exist positive integers $m = m(x, y) \geq 1$ and $n = n(x, y) \geq 1$ such that $xy = y^m p(yx) y^n$.

Theorem 4.2.1. Let R be a $d-g$ near ring satisfying either of the conditions $(I) - (V^*)$. Then R is periodic and commutative. Moreover, $R = P \uplus N$, where N is a subnear ring with trivial multiplication and P, the set of potent elements is a subring which itself is a ring.

In order to develop the proof of the above theorem, we require the follwoing lemmas which are proved in [22], [27], [32] and [38] respectively.

Lemma 4.2.1. Let R be a zero-symmetric near ring satisfying the properties :

(i) For each x in R, there exists an integer $n(x) > 1$ such that $x^{n(x)} = x$.

(ii) Every non-trivial homomorphic image of R contains a nonzero central idempotent.

Then $(R, +)$ is abelian.

Lemma 4.2.2. If R is a $d-g$ near ring with nilpotent elements central, then the set of all nilpotent elements forms an ideal.

Lemma 4.2.3. Let R be a $d-g$ near ring such that for each $x \in R$, there exist a positive integer $n = n(x)$ and an element s in the subnear ring generated by x for which $x^n = x^n s$. If $N \subseteq Z(R)$, then R is periodic and commutative.

Lemma 4.2.4. Let R be a near ring in which idempotents are multiplicatively central. If e and f are any idempotents, then there exists an idempotent g such that $ge = e$ and $gf = f$.

Lemma 4.2.5. If R is a zero-commutative periodic near ring, then $R = P + N$.

Using the fact that $d - g$ near rings are zero-symmetric and arguing in the similar manner as we have done in the proof of the Theorem 3.2.1, we can obtain the following :

Lemma 4.2.6. Let R be a zero-symmetric near ring satisfying any one of the conditions (I) – (V^*). Then nilpotent elements annihilate R on both the sides.

Proof of Theorem 4.2.1. Using Lemma 4.2.6, we have $N \subseteq Z(R)$ and $N^2 = \{0\}$. Replacing y by x in either of the conditions (I) - (V^*), we get an element r in the subnear ring generated by x such that $x^2 = x^2 r$. Hence by Lemma 4.2.3, R is periodic and commutative. Thus in view of Lemma 4.2.5, every element $x \in R$ can be expressed in the form $x = a + u$, where $a \in P$ and $u \in N$.

Now we show that P is a subring. Let $a, b \in P$ and choose integers $p = p(a) > 1$ and $q = q(b) > 1$ such that $a^p = a$ and $b^q = b$. Let $r = (p-1)q - (p-2) = (q-1)p - (q-2)$. Then it is clear that $a^r = a$ and $b^r = b$. Note that $e = a^{r-1}$ and $f = b^{r-1}$ are idempotents with $ea = a$ and $fb = b$. Obviously, $ab = a^r b^r = (ab)^r$, hence $ab \in P$ for all a, b in P. Moreover, by Lemma 4.2.2, N is an ideal. Since R/N has the $x^l = x$ property we have an integer $j > 1$ such that

$$(4.2.1) \qquad (a - b)^j = a - b + u \ ; \ u \in N.$$

Using Lemma 4.2.4, for any pair of idempotents e and f we can choose an idempotent g for which $ge = e$ and $gf = f$. Therefore $ga = a$ and $gb = b$. Now multiplying (4.2.1) by g we have $(a - b)^j = a - b$ i.e., $a - b \in P$. Also by Lemma 4.2.1, $(P, +)$ is abelian. Hence P itself is a ring.

Next we prove that the expression $x = a + u$ is unique. Trivially, $P \cap N = \{0\}$. Let $a + u = b + v$, where $a, b \in P$ and $u, v \in N$. Then $a - b = v - u \in P \cap N = \{0\}$, which yields that $a = b$ and $v = u$. Hence $R = P \uplus N$. $\square$

4.3 STRUCTURE OF D-NEAR RINGS

The example 4.2.1 is sufficient to notice that the above theorem can not be extended for general near rings. Indeed, the representation of the element $c \in R$ is not unique ($a + u = b + v = c$). With a view to extending the mentioned theorem we define a wider class of near rings which Steve Ligh and J. Luh [99] called as D-near rings.

Definition 4.3.2 (D-Near Ring). A near ring R is called a D-near ring if every nonzero homomorphic image T of R satisfies the following conditions :

(i) T has a nonzero right distributive element.

(ii) The additive group $(T, +)$ of T is abelian implies that T is a ring.

Remark 4.3.1. Obviously, all distributive and $d - g$ near rings are examples of D-near rings. However, the following example demonstrates that the D-near rings are generalizations of $d-g$ near rings.

Example 4.3.2. Let $R = \{0, a, b, c, u, v\}$ with addition and multiplication defined as follows :

+	0	a	b	c	u	v
0	0	a	b	c	u	v
a	a	0	v	u	c	b
b	b	u	0	v	a	c
c	c	v	u	0	b	a
u	u	b	c	a	v	0
v	v	c	a	b	0	u

.	0	a	b	c	u	v
0	0	0	0	0	0	0
a	0	a	a	a	0	0
b	0	a	c	b	v	u
c	0	a	b	c	u	v
u	0	0	0	0	0	0
v	0	0	0	0	0	0

Then R is a D-near ring which has a unique left identity c but $uc = 0 = vc$. This shows that c is not a right identity. But by [98, Theorem 3.2] a unique left identity in a $d-g$ near ring must be a right identity. Hence R is not a $d-g$ near ring.

We consider the following conditions, where $p(x)$ denotes an element of the near ring R which is a finite sum of powers x^k and additive inverses of such powers for $k \geq 2$.

(i) For every pair of elements x, y in R, there exist positive integers $m = m(x, y) \geq 1$ and $n = n(x, y) \geq 1$ such that $xy = y^m x^n p(x)$.

(ii) For every pair of elements x, y in R, there exist positive integers $m = m(x, y) \geq 1$ and $n = n(x, y) \geq 1$ such that $xy = x^m y^n p(x)$.

(iii) For every pair of elements x, y in R, there exists a positive integer $n = n(x, y) \geq 1$ such that $xy = (xy)^n p(x)$.

(iv) For every pair of elements x, y in R, there exists a positive integer $n = n(x, y) \geq 1$ such that $xy = (yx)^n p(x)$.

(v) For every pair of elements x, y in R, there exist positive integers $m = m(x, y) \geq 1$ and $n = n(x, y) \geq 1$ such that $xy = y^m p(x) y^n$.

(vi) For every pair of elements x, y in R, there exist positive integers $m = m(x, y) \geq 1$ and $n = n(x, y) \geq 1$ such that $xy = x^m p(y) x^n$.

The following lemma has been borrowed from [32].

Lemma 4.3.7. Let R be a near ring with unity 1. Then for every $x \in R$, $< x > \ = x < 1, x >$.

Now we begin with proving the following theorem which is in fact, an extension of a theorem of Bell [32, Theorem 12].

Theorem 4.3.2. Let R be a D-near ring satisfying $x^n = x^n p(x)$, where $n = n(x)$ a positive integer and $p(x) \in < x >$. If $N \subseteq Z(R)$, then R/N is periodic and commutative.

To develop the proof of the above theorem, we first establish the following :

Lemma 4.3.8. Let R be a near ring in which every nilpotent element is central, then the set of all nilpotent elements must be an ideal.

Proof. Let $u \in N$ such that $u^n = 0$. Since $N \subseteq Z(R)$, $(ru)^n = r^n u^n = r^n 0 = 0$, for all $u \in N$ and $r \in R$. Hence $RN \subseteq N$.

Now, let $u_1, u_2 \in N$ such that $u_1^m = 0$ and $u_2^n = 0$. Since $N \subseteq Z(R)$, $(u_1 - u_2)^{m+n} = 0$, for all $u_1, u_2 \in N$, so that $u_1 - u_2 \in N$.

Next, we shall show that $(N, +)$ is a normal subgroup of $(R, +)$ and we do this by induction on the degree of nilpotence. Let $u \in N$ and $r \in R$. If $u^2 = 0$, for all $u \in N$, then

$$\begin{aligned}
(r + u - r)^2 &= (r + u - r)(r + u - r) \\
&= (r + u - r)r + (r + u - r)u - (r + u - r)r \\
&= (r + u - r)r + u(r + u - r) - (r + u - r)r \\
&= (r + u - r)r + ur + u^2 - ur - (r + u - r)r \\
&= 0.
\end{aligned}$$

This shows that $(r + u - r) \in N$.

Now suppose that $u^k = 0$, for $k > 2$. If $r + u - r$ is nilpotent with index of nilpotence less than k, then putting $x = (r + u - r)r$. we get

$$\begin{aligned}
(r + u - r)^2 &= (r + u - r)r + (r + u - r)u - (r + u - r)r \\
&= x + u(r + u - r) - x \\
&= x + ur + u^2 - ur - x \\
&= (x + ur) + u^2 - (x + ur).
\end{aligned}$$

Repeating in the same way, we get for some $y \in R$.

$$\begin{aligned}
(r + u - r)^{k-1} &= (y + ur) + (u^2)^{k-1} - (y + ur) \\
&= 0.
\end{aligned}$$

Thus $(r + u - r) \in N$. Hence N is normal subgroup of $(R, +)$. Arguing as above, we can also show that $(x+v)y - xy \in N$, for all $v \in N$ and $x, y \in R$. Hence N is an ideal of R.

Proof of Theorem 4.3.2. Since $N \subseteq Z(R)$, N is an ideal by Lemma 4.3.8. Consider the near ring R/N. Since R/N can be written as a subdirect product of near rings without zero divisors, we may assume R/N has no nonzero divisors of zero.

Further, R is given to be D-near ring and so in R/N, a homomorphic image of R, there exists a nonzero distributive element d. Note that in view of property $x^n = x^n p(x)$, $d^n = d^n t(d)$, for some $t(d) \in < d >$ i.e., $d^{n-1}(d - dt(d) = 0$, for some $t(d) \in < d >$ which yields that $d = dt(d)$.

Now we assert that $e = t(d)$ is an idempotent in R/N. Indeed,

$$\begin{aligned}
e^2 &= t(d)t(d) \\
&= (d + d^2 + \dots\dots + d^s - d - d^2 - \dots\dots - d^s)t(d) \\
&= dt(d) + d^2 t(d) + \dots\dots + d^s t(d) - dt(d) - d^2 t(d) - \dots\dots - d^s t(d) \\
&= d + d^2 + \dots\dots + d^s - d - d^2 - \dots\dots - d^s \\
&= e.
\end{aligned}$$

Since $er - er = e(er - r) = 0$, for every $r \in R/N$, $er = r$. Thus e is a left identity in R/N.

Considering arbitrary elements $x, y \in R/N$ and using the fact that e commutes with d and $d = dt(d) = de$, we have

$$0 = (x+y)de - (xde + yde) = (x+y)ed - (xed + yed)$$

$$= (x+y)ed - (xe+ye)d = [(x+y)e - (xe+ye)]d.$$

Again since R/N has no nonzero divisors of zero, e is a distributive element of R/N. Hence e is the multiplicative identity in R/N.

Now, let x be an arbitrary nonzero element of R/N. Then $x = xt(x)$, where $t(x) \in < x >$. Using Lemma 4.3.7, we can write $t(x) = xt'(x)$ for some $t'(x) \in < 1, x >$. Now $x = x^2 t'(x)$ i.e., $x(1 - xt'(x)) = 0$, which yields that $xt'(x) = 1$. Thus R/N is a division near ring and hence additively commutative by Theorem 1.4.5, so R/N is a ring. Using Chacron's criterion [52], R/N is periodic and a commutative ring by [28, Theorem 2]. □

The proof of the above theorem runs on the parallel lines, if we replace property $x^n = x^n p(x)$ by $x^n = x^n p(x) x^n$.

Theorem 4.3.3. Let R be a near ring satisfying either of the conditions (i) - (iv). Then $RN = NR = \{0\}$.

Proof. Notice that R satisfying condition (i) is zero-commutative. Indeed, if $xy = 0$, then there exist positive integers $m' = m'(y,x) \geq 1$ and $n' = n'(y,x) \geq 1$, such that $yx = x^{m'} y^{n'} p(y) = 0$. Replacing y by x in (i), we find that

$$(4.3.1) \qquad x^2 = x^{r+s} h(x) \ ; \ \text{for } r+s \geq 2.$$

If $u \in N$, then by repeated use of (4.3.1), we get $u^2 = 0$. Now for any $u \in N$ by condition (i), we have $ux = x^l u^t g(u) = 0$, for $l = l(u,x) \geq 1$, $t = t(u,x) \geq 1$. Zero-commutativity in R yields that $xu = 0$, for $u \in N$ and $x \in R$. Hence

$$NR = RN = \{0\}. \tag{4.3.2}$$

The proof of the lemma follows similarly for R satisfying any one of the conditions (ii) - (iv). $\Box$

Proceeding on the same lines, we can prove the following :

Theorem 4.3.4. Let R be a zero-symmetric near ring satisfying condition (v). Then $RN = NR = \{0\}$.

Theorem 4.3.5. Let R be a zero-commutative near ring satisfying condition (vi). Then $RN = NR = \{0\}$.

Before establishing our structure theorem, we pause to prove the following lemma :

Lemma 4.3.9. Let R be a D-near ring satisfying either of the conditions (i) and (iv). Then idempotent elements of R are central.

Proof. Let e be an idempotent and $x \in R$. Then by condition (i) there exist positive integers $l' = l'(x, e) \geq 1$ and $t' = t'(x, e) \geq 1$ such that $xe = ex^{t'}p(x)$. Multiplying by e on the left, we get $exe = xe$. Application of Theorem 4.3.3, yields that $C(R) \subseteq N \subseteq Z(R)$ and we have $e(xe - ex) = 0$, for all $x \in R$. Hence $ex = xe$, for all $x \in R$. $\Box$

Theorem 4.3.6. Let R be a D-near ring satisfying either of the conditions (i) and (iv). Then $R = P \uplus N$, where N is a subnear ring with trivial multiplication and P, the set of potent elements is a subring which itself is a ring.

Proof . In view of Theorem 4.3.2, for each $x \in R$, there exist distinct positive integers $m = m(x)$ and $n = n(x)$ such that $x^m - x^n \in N$. Hence using Theorem 4.3.3, we have $x^{m+1} = x^{n+1}$, for each $x \in R$ and R is periodic.

Now we show that P is a subring. Let $a, b \in P$ and choose integers $p = p(a) > 1$ and $q = q(b) > 1$ such that $a^p = a$ and $b^q = b$. Let $r = (p-1)q - (p-2) = (q-1)p - (q-2)$. Then it is clear that $a^r = a$ and $b^r = b$. Note that $e = a^{r-1}$ and $f = b^{r-1}$ are idempotents with $ea = a$ and $fb = b$. By using Theorem 4.3.2 and Theorem 4.3.3, $C(R) \subseteq N \subseteq Z(R)$ and $a^2b = aba$, for all, $a, b \in P$. Obviously, $ab = a^r b^r = (ab)^r$, hence $ab \in P$, for all $a, b \in P$. Moreover, since R/N has the $x^l = x$ property we have an integer $j > 1$ such that

$$(a - b)^j = a - b + u \ ; \ u \in N. \tag{4.3.3}$$

Using Lemma 4.2.4, we can choose an idempotent g for which $ge = e$ and $gf = f$. Therefore, $ga = a$ and $gb = b$. Now multiplying (4.3.3) by g we have $(a-b)^j = a - b$ i.e. $a - b \in P$ yielding that P is a subnear ring. Also by Lemma 4.2.1, $(P, +)$ is abelian. Hence P itself is a ring.

Trivially, $P \cap N = \{0\}$. Let $a + u = b + v$, where $a, b \in P$ and $u, v \in N$. Then $a - b = v - u \in P \cap N = \{0\}$, which yields that $a = b$ and $v = u$. Hence $R = P \uplus N$. □

Remark 4.3.2. The following example shows that Theorem 4.3.6 can not be extended to arbitrary near rings.

Example 4.3.3. Let $R = \{0, a, b, c\}$ with addition and multiplication obtained as follows :

+	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

.	0	a	b	c
0	0	0	0	0
a	0	a	0	a
b	0	0	0	0
c	0	c	0	c

It is easy to notice that R is a near ring satisfying the condition $xy = xy^2x$, for all $x, y \in R$. But $P = \{0,\ a,\ c\}$ is not a subring of R. Thus we can not get an orthogonal sum decompositon of a near ring R satisfying the condition (vi).

However, under some extra hypotheses either of the above conditions guarantees an orthogonal sum decomposition of near rings. In this direction, we state the following theorems which can be proved using the techniques as those employed in the proof of Theorem 4.3.6.

Theorem 4.3.7. Let R be a D-near ring satisfying either of the conditions (ii) and (iii). Moreover, if idempotent elements of R are central, then $R = P \uplus N$, where N is a subnear ring with trivial multiplication and P, the set of potent elements is a subring which itself is a ring.

Theorem 4.3.8. Let R be a zero-symmetric D-near ring satisfying condition (v). Moreover, if idempotent elements of R are central, then $R = P \uplus N$, where N is a subnear ring with trivial multiplication and P, the set of potent elements is a subring which itself is a ring.

Theorem 4.3.9. Let R be a zero-commutative D-near ring satisfying condition (vi). Moreover, if idempotent elements of R are central, then $R = P \uplus N$, where N is a subnear ring with trivial multiplication and P, the set of potent elements is a subring which itself is a ring.

4.4 SOME CONDITIONS UNDER WHICH NEAR RINGS ARE RINGS

Part of the recent work on near rings has been concerned with sufficient conditions for near rings to be rings. Ligh [97], by proving that distributively generated Boolean near rings are rings has conjectured the possibility that some of the more complicated polynomial conditions may turn distributively generated near rings into rings. Later many authors including Bell worked on this line (see for example [23], [111] and [112]). Motivated by these results one may jump to the conclusion that under many polynomial identity conditions implying commutativity in rings, the structure of $d-g$ near rings should coincide to that of rings. However, Example 4.2.1 illustrates that in general this need not be true.

In this section, we continue the study considering d-g near rings satisfying conditions $(I) - (V^*)$ discussed in section 4.2.

Theorem 4.4.10. Let R be a $d-g$ near ring satisfying any one of the conditions $(I) - (V^*)$. If $R^2 = R$, then R is a commutative ring.

Proof. Since R is commutative by Theorem 4.2.1, for any $x, y, z \in R$, we have

$$(x+y)z = z(x+y) = zx + zy = xz + yz.$$

This implies that R is distributive. Thus for any $w, x, y, z \in R$,

$$(x+y)(z+w) = (x+y)z + (x+y)w = x(z+w) + y(z+w).$$

Now we claim that R^2 is additively commutative. Indeed, for $w, x, y, z \in R$, we have

$$\begin{aligned} yz + xw &= -xz + (xz + yz + xw + yw) - yw \\ &= -xz + ((x+y)z + (x+y)w) - yw \\ &= -xz + (x(z+w) + y(z+w)) - yw \\ &= -xz + (xz + xw + yz + yw) - yw \\ &= xw + yz. \end{aligned}$$

Hence R^2 is additively commutative. Since $R^2 = R$, we find that R is also additively commutative. Hence R is a commutative ring. □

Again in view of Theorem 4.2.1 and using Theorem 1.4.3, we conclude,

Theorem 4.4.11. Let R be a $d-g$ near ring with unity 1 satisfying any one of the conditions $(I) - (V^*)$. Then R is a commutative ring.

Chapter-5

COMMUTATIVITY OF RINGS WITH GENERALIZED DERIVATIONS

5.1 INTRODUCTION

Let R_1 and R_2 be rings (need not be commutative and may not be containing unities). An additive mapping $d : R_1 \longrightarrow R_2$ is said to be a derivation from R_1 to R_2 if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R_1$. The study of derivations though initiated long back, got impetus only after E. C. Posner [110] established in 1957 two very striking results which attracted a wide circle of mathematicians. The results under reference state that (i) Let R be a prime ring of characteristic different from 2 and d_1, d_2 derivations of R such that the iterate d_1d_2 is also a derivation. Then at least one of d_1, d_2 must be zero, (ii) if d is a derivation of a prime ring R such that for every element $x \in R$, $[x, d(x)] \in Z(R)$, then either the ring is commutative or d is zero.

The recent literature includes several results on commutativity in prime and semiprime rings with commutator constraints involving elements of the ring and their images under suitable maps (see [49], for example, where more references can be found). There is also a growing interest on commutativity preserving maps (to mention a few [11], [34], [49], [63]). *A mapping $f : R \longrightarrow R$ is called commutativity preserving on a subset S of a ring R if $[x, y] = 0$ implies that $[f(x), f(y)] = 0$, for all $x, y \in S$. The mapping f is called strong commutativity preserving on S if $[f(x), f(y)] = [x, y]$, for all $x, y \in S$.*

The present chapter deals with same type of work. In section 5.2, we give some basic concepts about the subject and present some preliminary results. Though some of these results are already proved in different papers, we prefer to give the outlines of their proofs to acquaint the reader with the techniques generally used in dealing with derivations. Of course, at places proofs will be found to be fresh, filling the gaps in earlier proofs. This section may, in fact, be treated as a warm-up exercise for subsequent material.

The notion of derivation is generalized in various directions such as left derivation, (θ, ϕ)-derivation, semi-derivation, generalized derivation, Jordan derivation and Lie derivation etcetera. Very recently, Hvala [82] initiated the algebraic study of generalized derivation and extended some results concerning derivations to generalized derivations (see Definition 5.3.1). In section 5.3, we discuss the commutativity of a prime ring R admitting a generalized derivation F with associated derivation d satisfying any one of the following conditions: (i) $[d(x), F(y)] = [x, y]$, (ii) $[d(x), F(y)] + [x, y] = 0$, (iii) $[d(x), F(y)] = 0$, (iv) $d(x) \circ F(y) = 0$, (v) $d(x) \circ F(y) = x \circ y$, (vi) $d(x) \circ F(y) + x \circ y = 0$, (vii) $d(x)F(y) - xy \in Z(R)$ and (viii) $d(x)F(y) + xy \in Z(R)$, for all x, y in some appropriate subset of the ring R.

Recently, Ashraf and Nadeem [14] proved that a prime ring R with a nonzero ideal I must be commuatative, if it admits a derivation d satisfying either of the properties : (i) $d(xy) + xy \in Z(R)$ and (ii) $d(xy) - xy \in Z(R)$, for all $x, y \in I$. We have succeeded in establishing the above result for generalized derivation in prime rings involving a Lie ideal which has been included in section 5.4.

5.2 SOME PRELIMINARY DEFINITIONS AND RESULTS

We begin with some preliminary definitions and results in order to make our subject matter as much self contained as possible.

Perhaps, motivated by two basic properties of a differential operator, say D (namely, (i) $D(f_1+f_2) = D(f_1)+D(f_2)$, (ii) $D(f_1f_2) = D(f_1)f_2+f_1D(f_2)$), the notion of derivations was introduced in rings.

Definition 5.2.1 (Derivation). A mapping $d : R_1 \longrightarrow R_2$ from a ring R_1 to a ring R_2 is said to be a derivation if for all $x, y \in R_1$, the following hold :

(i) $d(x_1 + x_2) = d(x_1) + d(x_2)$,

(ii) $d(x_1x_2) = d(x_1)x_2 + x_1d(x_2)$

Example 5.2.1. Let $R = I\!R[X]$ be the ring of polynomials over the field $I\!R$ of real numbers and

$$p(x) = a_0 + a_1x + a_2x^2 + \dots\dots\dots + a_nx^n, \ a_i \in I\!R$$

be an arbitrary element. Set

$$d(p(x)) = a_1 + 2a_2x + \dots\dots\dots + na_nx^{n-1}.$$

It can be readily verified that d is a derivation on R.

Example 5.2.2. Let R be a ring and a be a fixed element of R. Define the mapping $\delta : R \longrightarrow R$ by $\delta(x) = [x, a] = xa - ax$, for all $x \in R$, then δ is a derivation on R which is usually called the inner derivation of R and generally denoted by I_a.

Example 5.2.3. Let R be the ring of 2×2 matrices over $GF(2)$. Define $d : R \longrightarrow R$ by $d\begin{pmatrix} a\,b \\ c\,d \end{pmatrix} = \begin{pmatrix} 0 & -b \\ c & 0 \end{pmatrix}$; $a, b, c, d \in GF(2)$. It is easy to see that d is a derivation on R.

We shall make frequent use of the following well-known results which may be found in [40], [104], [13] and [116] respectively. However, in order to make our subsequent text as much self contained as possible, we are giving the sketches of their proofs that are at times simpler and straightforward.

Proposition 5.2.1. Let I be a nonzero left (right) ideal of a prime ring R. If d is a nonzero derivation of R, then d is nonzero on I.

Proof. Let $d(x) = 0$, for all $x \in I$. Replacing x by rx in the above relation and using it, we get $d(r)x = 0$, for all $x \in I$ and $r \in R$. Now replace x by sx, to get $d(r)sx = 0$, for all $x \in I$ and $r, s \in R$ i.e., $d(r)RI = \{0\}$, for all $r \in R$. Since I is nonzero, the primeness of R yields that $d(r) = 0$, for all $r \in R$. $\square$

Proposition 5.2.2. If the prime ring R contains a nonzero commutative right ideal A, then R is commutative.

Proof. Since I is commutative, $I_x(A) = [x, A] = \{0\}$, for all $x \in A$. By the above proposition, $I_x = 0$ on R and x is in the centre. Thus $[x, R] = \{0\}$, for every $x \in A$. Hence $I_a(A) = \{0\}$, for all $a \in R$. Again using the above proposition, we obtain $I_a = 0$ and a is in the centre for all $a \in R$. Therefore, R is commutative. $\square$

Proposition 5.2.3. Let R be a 2-torsion free prime ring and I be a nonzero ideal of R. If R admits a derivation d with $d^2(x) = 0$, for all $x \in I$, then $d = 0$.

Proof. we have $d^2(x) = 0$, for all $x \in I$. Replacing x by xy, we get $d^2(xy) = 0$, for all $x, y \in I$ i.e.,

$$d^2(x)y + 2d(x)d(y) + xd^2(y) = 0, \text{ for all } x, y \in I.$$

But $d^2(x) = 0 = d^2(y) = 0$ by the hypothesis, the above relation implies that $2d(x)d(y) = 0$, for all $x, y \in I$. Since R is 2-torsion free, we find that $d(x)d(y) = 0$, for all $x, y \in I$. Now for any $r \in R$, replace y by yr, to get $d(x)yd(r) = 0$, for all $x, y \in I$ and hence $d(x)IRd(r) = 0$, for all $x, \in I$ and $r \in R$. Thus primeness of R yields that either $d(r) = 0$ or $d(x)I = \{0\}$. If $d(x)I = \{0\}$, for all $x \in I$, then $d(x)RI = \{0\}$, for all $x \in I$. Since R is prime and $I \neq \{0\}$, we find that $d(x) = 0$, for all $x \in I$ and by Proposition 5.2.1, we get the required result. □

proposition 5.2.4. Let R be a 2-torsion free prime ring and U be a nonzero Lie ideal of R. If U is a commutative Lie ideal of R, then $U \subseteq Z(R)$, the centre of R.

Proof. Since U is a commutative Lie ideal of R, $[u, v] = 0$, for all $u, v \in U$. Replacing v by $[u, r]$ in the above relation, we get $[u, [u, r]] = 0$, for all $u \in U$ and $r \in R$. Again replace r by rs, to get $[u, [u, rs]] = 0$, for all $u \in U$ and $r, s \in R$ that is,

$$[u, [u, r]]s + r[u, [u, s]] + 2[u, r][u, s] = 0, \text{ for all } u \in U \text{ and } r, s \in R.$$

This implies that $2[u, r][u, s] = 0$, for all $u \in U$ and $r, s \in R$. Since R is 2-torsion free, we find that $[u, r][u, s] = 0$, for all $u \in U$ and $r, s \in R$. Replacing s by sr, we get $[u, r]s[u, r] = 0$, for all $u \in U$ and $r, s \in R$ i.e., $[u, r]R[u, r] = \{0\}$, for all $u \in U$, $r \in R$. Thus primeness of R implies that $[u, r] = 0$, for all $u \in U$,

$r \in R$ and hence $U \subseteq Z(R)$. $\square$

Remark 5.3.1. The above result can be extended to semiprime rings also.

5.3 IDEALS AND GENERALIZED DERIVATIONS IN PRIME RINGS

The notion of derivation in rings has been generalized in several directions such as left derivation, Jordan derivation, semi-derivation ; to mention a few. In the theory of operator algebras, an additive mapping $D_{a,b} : A \longrightarrow A$ on an algebra A defind by $D_{a,b}(x) = ax + xb$, for fixed $a, b \in A$ plays an important role. One can notice that such a map can be considered as a generalization of inner derivation $I_a : x \longrightarrow ax - xa$ and is naturally termed as *generalized inner derivation* or alternatively *inner generalized derivation*. Further, for any $x, y \in A$

$$\begin{aligned} D_{a,b}(xy) &= a(xy) + (xy)b \\ &= (ax + xb)y - xby + (xy)b \\ &= D_{a,b}(x)y + x(yb - by) \\ &= D_{a,b}(x)y + xI_b(y). \end{aligned}$$

This prompts us to formulate the following definition.

Definition 5.3.2 (Generalized Derivation). Let S be a nonempty subset of R. An additive mapping $F : R \longrightarrow R$ is said to be a generalized derivation on S if there exists a derivation $d : R \longrightarrow R$ such that $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in S$.

Example 5.3.4. Let $R=\left\{\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a, b \in \mathbb{Z}\right\}$. Define $F : R \longrightarrow R$ by $F(x) = 2cx - xc$, where $c = e_{12} + e_{21}$. Then F is a generalized derivation with associated derivation d given by $d(x) = cx - xc$.

As obseved in [86] that the concept of generalized derivation includes both the concept of derivation as well as that of inner generalized derivation. Further, with $d = 0$, generalized derivation leads to the concept of left multiplier.

Remark 5.3.2. The following example is sufficient to show that a generalized derivation need not be a derivation in general.

Example 5.3.5. Let $R = \left\{\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b \in \mathbb{Z}_2\right\}$. Define a map $F : R \longrightarrow R$ by $F\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$ and a derivation $d : R \longrightarrow R$ by $d\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$. Then it can be easily verified that F is a generalized deivation on R but not a derivation on R.

Over the past few years there has been an ongoing interest in studying the relationship between the commutativity of a ring and the existence of certain specific types of derivations in the ring. In the present section, we discuss the commutativity of prime rings admitting a generalized derivation which satisfies certain functional identities.

Theorem 5.3.1. Let R be a 2-torsion free prime ring and I be a nonzero ideal of R. If R admits a generalized derivation F with associated derivation d such that $[d(x), F(y)] = [x, y]$, for all $x, y \in I$, then either $d = 0$ or R is commutative.

For developing the proof of the above theorem, we require the following lemma which can be found in [41].

Lemma 5.3.1. Let R be a prime ring and I be a nonzero left ideal of R. If R admits a nonzero derivation d such that $[x, d(x)]$ is central for all $x \in I$, then R is commutative.

Proof of Theorem 5.3.1. If $F = 0$, then $[x, y] = 0$, for all $x, y \in I$ and so by Proposition 5.2.2, R is commutative. Hence onward, we assume that $F \neq 0$. We have

$$(5.3.1) \qquad [d(x), F(y)] = [x, y], \quad \text{for all } x, y \in I.$$

Replacing y by yz in (5.3.1) and using (5.3.1), we have

$$(5.3.2) \qquad F(y)[d(x), z] + y[d(x), d(z)] + [d(x), y]d(z) = y[x, z], \text{ for all } x, y, z \in I.$$

Again replacing z by $zd(x)$ in (5.3.2) and using (5.3.2), we obtain

$$(5.3.3) \qquad y[d(x), z]d^2(x) + yz[d(x), d^2(x)] + [d(x), y]zd^2(x) = yz[x, d(x)].$$

Now replace y by ry in (5.3.3), to get

$$ryz[d(x), d^2(x)] + ry[d(x), z]d^2(x) + r[d(x), y]zd^2(x) + [d(x), r]yzd^2(x) = ryz[x, d(x)].$$

Then (5.3.3) yields that $[d(x), r]yzd^2(x) = 0$, for all $x, y, z \in I$, $r \in R$ i.e., $[d(x), r]IRd^2(x) = \{0\}$, for all $x \in I$ and $r \in R$. The primeness of R forces that for each fixed $x \in I$, either $[d(x), r]I = \{0\}$ or $d^2(x) = 0$. Now let $I_1 = \{x \in I | \ d^2(x) = 0\}$ and $I_2 = \{x \in I | \ [d(x), r]I = \{0\}, \text{ for all } r \in R\}$. Then I_1 and I_2 are additive subgroups of I whose union is I. But a group can not be union of two of its proper subgroups and hence $I = I_1$ or $I = I_2$. If $I = I_1$, then $d^2(x) = 0$, for all $x \in I$. Thus by Proposition 5.2.3, we get $d = 0$. On the other hand, if $I = I_2$, then $[d(x), r]I = \{0\}$, for all $x \in I$, $r \in R$ and hence $[d(x), r]RI = \{0\}$, for all $x \in I$, $r \in R$. Since R is prime and $I \neq \{0\}$, it follows that $[d(x), r] = 0$, for all $x \in I$, $r \in R$. In particular, $[d(x), x] = 0$, for all $x \in I$. Hence R is commutative by Lemma 5.3.1. □

Proceeding on the same lines with necessary variations, we can prove the following :

Theorem 5.3.2. Let R be a 2-torsion free prime ring and I be a nonzero ideal of R. If R admits a generalized derivation F with associated derivation d such that $[d(x), F(y)] + [x, y] = 0$, for all $x, y \in I$, then either $d = 0$ or R is commutative.

Theorem 5.3.3. Let R be a 2-torsion free prime ring and I be a nonzero ideal of R. If R admits a generalized derivation F with associated derivation d such that $[d(x), F(y)] = 0$, for all $x, y \in I$, then either $d = 0$ or R is commutative.

Proof. We have

$$(5.3.4) \qquad [d(x), F(y)] = 0, \quad \text{for all } x, y \in I.$$

Replacing y by yz in (5.3.4) and using (5.3.4), we get

$$(5.3.5) \qquad F(y)[d(x), z] + y[d(x), d(z)] + [d(x), y]d(z) = 0, \quad \text{for all } x, y, z \in I.$$

Now replacing z *by* $zd(x)$ in (5.3.5) and using (5.3.5), we obtain

(5.3.6) $yz[d(x), d^2(x)] + y[d(x), z]d^2(x) + [d(x), y]zd^2(x) = 0,$ for all $x, y, z \in I$.

Again replace y by ry in (5.3.6) and use (5.3.6), to get $[d(x), r]yzd^2(x) = 0$, for all $x, y, z \in I$ and $r \in R$ i.e., $[d(x), r]IRd^2(x) = \{0\}$, for all $x \in I$, $r \in R$. Now using the similar arguments as we have used in the proof of Theorem 5.3.1, we get the required result. □

Theorem 5.3.4. Let R be a 2-torsion free prime ring and I be a nonzero ideal of R. If R admits a generalized derivation F with associated derivation d such that $d(x) \circ F(y) = 0$, for all $x, y \in I$, then either $d = 0$ or R is commutative.

Proof. We have $d(x) \circ F(y) = 0$, for all $x, y \in I$. Replacing y by yr, we get $d(x) \circ F(yr) = 0$, for all $x, y \in I$ and $r \in R$ and hence we find that

$$(d(x) \circ y)d(r) - y[d(x), d(r)] + (d(x) \circ F(y))r - F(y)[d(x), r] = 0.$$

Now using our hypotheses, the above relation yields that

$$(d(x) \circ y)d(r) - y[d(x), d(r)] - F(y)[d(x), r] = 0, \quad \text{for all } x, y \in I \text{ and } r \in R.$$

Again replace r by $d(x)$, to get

(5.3.7) $\quad (d(x) \circ y)d^2(x) - y[d(x), d^2(x)] = 0, \quad$ for all $x, y \in I$ and $r \in R$.

Now replacing y by zy in (5.3.7), we obtain

$$(d(x) \circ zy)d^2(x) - zy[d(x), d^2(x)] = 0, \quad \text{for all } x, y, z \in I.$$

This implies that

$$z(d(x) \circ y)d^2(x) + [d(x), z]yd^2(x) - zy[d(x), d^2(x)] = 0, \quad \text{for all } x, y, z \in I.$$

In view of (5.3.7), the above expression yields that $[d(x), z]yd^2(x) = 0$, for all $x, y, z \in I$ and hence $[d(x), z]IRd^2(x) = \{0\}$, for all $x, z \in I$. Further, application of similar arguments as used in the end of the proof of Theorem 5.3.1, we get the required result. □

Theorem 5.3.5. Let R be a 2-torsion free prime ring and I be a nonzero ideal of R. If R admits a generalized derivation F with associated derivation d such that $d(x) \circ F(y) = x \circ y$, for all $x, y \in I$, then either $d = 0$ or R is commutative.

Proof. Given that $d(x) \circ F(y) = x \circ y$, for all $x, y \in I$. If $F = 0$, then $x \circ y = 0$, for all $x, y \in I$. Replacing y by yz and using the fact that $x \circ y = 0$, we obtain $y[x, z] = 0$, for all $x, z \in I$. In particular, $IR[x, z] = \{0\}$, for all $x, z \in I$ and the primeness of R forces that $[x, z] = 0$, for all $x, z \in I$. Hence R is commutative by Proposition 5.2.2.

Therefore, now onward we assume that $F \neq 0$. For any $x, y \in I$, we have $d(x) \circ F(y) = x \circ y$. Replacing y by yr, we get

$$(d(x) \circ y)d(r) - y[d(x), d(r)] + (d(x) \circ F(y))r - F(y)[d(x), r] = (x \circ y)r - y[x, r].$$

Using our hypotheses, we get

$$(d(x) \circ y)d(r) - y[d(x), d(r)] - F(y)[d(x), r] + y[x, r] = 0.$$

In the above expression replacing r by $d(x)$, we obtain

$$(5.3.8) \qquad (d(x) \circ y)d^2(x) - y[d(x), d^2(x)] + y[x, d(x)] = 0, \quad \text{for all } x, y \in I.$$

Now replacing y by zy in (4.2.8), we get

$$(z(d(x) \circ y) + [d(x), z]y)d^2(x) - zy[d(x), d^2(x)] + zy[x, d(x)] = 0.$$

In view of (5.3.8), we find that $[d(x), z]yd^2(x) = 0$, for all $x, y, z \in I$ and hence $[d(x), z]IRd^2(x) = \{0\}$, for all $x, z \in I$. Thus primeness of R forces that either $d^2(x) = 0$ or $[d(x), z]I = \{0\}$. Arguing in the similar manner as we have done in the proof of Theorem 5.3.1, we find the required result. □

Using the similar arguments as we have used in the above theorem, we can prove the following which generalizes Theorem 4.5 of [13].

Theorem 5.3.6. Let R be a 2-torsion free prime ring and I be a nonzero ideal of R. If R admits a generalized derivation F with associated derivation d such that $d(x) \circ F(y) + x \circ y = 0$, for all $x, y \in I$, then either $d = 0$ or R is commutative.

Theorem 5.3.7. Let R be a prime ring and I be a nonzero ideal of R. If R admits a generalized derivation F with associated derivation d such that $d(x)F(y) - xy \in Z(R)$, for all $x, y \in I$, then either $d = 0$ or R is commutative.

Proof. For any $x, y \in I$, we have $d(x)F(y) - xy \in Z(R)$. If $F = 0$, then $xy \in Z(R)$, for all $x, y \in I$. In particular, $[xy, x] = 0$, for all $x, y \in I$ and hence $x[y, x] = 0$. Replacing y by yz, we get $xy[z, x] = 0$, for all $x, y, z \in I$. Further, replace y by ry, to get $xry[z, x] = 0$, for all $x, y, z \in I$, $r \in R$ and hence $xRI[z, x] = \{0\}$. Thus primeness of R forces that for each $x \in I$, either $x = 0$ or $I[z, x] = \{0\}$. But $x = 0$ also implies that $I[z, x] = \{0\}$ and hence $IR[z, x] = \{0\}$, for all $x, z \in I$. Since $I \neq \{0\}$ and R is prime, the above relation yields that $[z, x] = 0$, for all $x, z \in I$. Hence by Proposition 5.2.2, R is commutative.

Now, we assume that $F \neq 0$. For any $x, y \in I$, we have

$d(x)F(y) - xy \in Z(R)$. Replacing y by yr, we get $d(x)F(yr) - xyr \in Z(R)$, for all $x, y \in I$ and $r \in R$ i.e.,

$$(d(x)F(y) - xy)r + d(x)yd(r) \in Z(R), \quad \text{for all } x, y \in I \text{ and } r \in R.$$

This implies that

$$[d(x)yd(r),\ r] \ = \ 0, \quad \text{for all } x, y \in I \text{ and } r \in R.$$

Hence it follows that

$$d(x)[yd(r), r] + [d(x), r]yd(r) \ = \ 0, \quad \text{for all } x, y \in I \text{ and } r \in R.$$

Now replacing y by $d(x)y$, we get $[d(x), r]d(x)yd(r) = 0$, for all $x, y \in I$ and $r \in R$. This implies that $[d(x), r]d(x)r_1yd(r) = 0$, for all $x, y \in I$, $r, r_1 \in R$ and hence $[d(x),\ r]d(x)RId(r) = \{0\}$, for all $x \in I$, $r \in R$. Thus for each $r \in R$, primeness of R forces that either $[d(x),\ r]d(x) = 0$ or $Id(r) = \{0\}$. Now if $A = \{r \in R \mid [d(x), r]d(x) = 0, \text{ for all } x \in I\}$, $B = \{r \in R \mid Id(r) = \{0\}\ \}$, then by using Braur's trick, we find that either $[d(x), r]d(x) = 0$ or $Id(r) = \{0\}$. If $Id(r) = \{0\}$, for all $r \in R$, then $IRd(r) = \{0\}$, for all $r \in R$. Since R is prime and $I \neq \{0\}$, the above relation yields that $d = 0$. On the other hand, assume the remaining possibility that $[d(x), r]d(x) = 0$, for all $x \in I$ and $r \in R$. For any $s \in R$, replace r by rs, to get $[d(x), r]sd(x) = 0$, for all $x \in I$, $r \in R$ and hence $[d(x), r]Rd(x) = \{0\}$, for all $x \in I$, $r \in R$. The primeness of R implies that for each $x \in I$, either $d(x) = 0$ or $[d(x), r] = 0$. But $d(x) = 0$ also implies that $[d(x), r] = 0$. Hence $[d(x), r] = 0$, for all $x \in I$, $r \in R$ and by Lemma 5.3.1, R is commutative. □

Proceeding on the same lines with necessary variations, we can prove the following theorem which includes Theorem 2.5 of [14].

Theorem 5.3.8. Let R be a prime ring and I be a nonzero ideal of R. If R admits a generalized derivation F with associated derivation d such that $d(x)F(y) + xy \in Z(R)$, for all $x, y \in I$, then either $d = 0$ or R is commutative.

Theorem 5.3.9. Let R be a prime ring and I be a nonzero ideal of R. Then the followings are equivalent:

(i) R is 2-torsion free and R admits a generalized derivation F with associated derivation $d \neq 0$ such that $[d(x), F(y)] - [x, y] = 0$ or $[d(x), F(y)] + [x, y] = 0$, for all $x, y \in I$.

(ii) R admits a generalized derivation F with associated derivation $d \neq 0$ such that $d(x)F(y) - xy \in Z(R)$ or $d(x)F(y) + xy \in Z(R)$, for all $x, y \in I$.

(iii) R is commutative.

Proof. Obviously, $(iii) \Rightarrow (i)$ and $(iii) \Rightarrow (ii)$.

$(i) \Rightarrow (iii)$. For each fixed $x \in I$, we put $I_1 = \{y \in I \mid [d(x), F(y)] - [x, y] = 0\}$ and $I_2 = \{y \in I \mid [d(x), F(y)] + [x, y] = 0\}$. Then it can be easily seen that I_1 and I_2 both are additive subgroups of I whose union is I. Then by Braur's trick, either $I_1 = I$ or $I_2 = I$. Further using similar arguments as above, we find that $I = \{x \in I \mid I_1 = I\}$ or $I = \{x \in I \mid I_2 = I\}$. Therefore, R is commutative by Theorems 5.3.1 and 5.3.2.

$(ii) \Rightarrow (iii)$. For each fixed $x \in I$, we put $I_1 = \{y \in I \mid d(x)F(y) - xy \in Z(R)\}$ and $I_2 = \{y \in I \mid d(x)F(y) + xy \in Z(R)\}$. Using the similar arguments as above and using Theorems 5.3.7 and 5.3.8, we get the required result.

Remark 5.3.3. The following example demonstrates that the above results are not true in case of arbitrary rings.

Example 5.3.6. Consider S as any ring. Let $R=\left\{\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a, b \in S\right\}$ and let $I=\left\{\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \mid b \in S\right\}$ be an ideal of R. Define $F : R \longrightarrow R$ by $F(x) = 2e_{11}x - xe_{11}$, where e_{ij} denotes an identity matrix. Then F is a generalized derivation with associated derivation d given by $d(x) = e_{11}x - xe_{11}$. It can be easily seen that R satisfies the properties: (i) $d(x) \circ F(y) = 0$, (ii) $[d(x), F(y)] = 0$, (iii) $d(x) \circ F(y) = x \circ y$, (iv) $d(x) \circ F(y) + x \circ y = 0$, (v) $d(x)F(y) - xy \in Z(R)$, (vi) $d(x)F(y) + xy \in Z(R)$, (vii) $[d(x), F(y)] = [x, y]$ and (viii) $[d(x), F(y)] + [x, y] = 0$, for all $x, y \in I$. However, R is not commutative.

5.4 LIE IDEALS AND GENERALIZED DERIVATIONS IN PRIME RINGS

In [14], Ashraf and Nadeem established that a prime ring R with a nonzero ideal I must be commuatative, if it admits a derivation d satisfying either of the properties : (i) $d(xy) + xy \in Z(R)$ and (ii) $d(xy) - xy \in Z(R)$, for all $x, y \in I$. Inspired by this result, we have proved the following :

Theorem 5.4.10. Let R be a 2-torsion free prime ring and U be a nonzero Lie ideal of R with $u^2 \in U$, for all $u \in U$. If R admits a generalized derivation F with associated derivation $d \neq 0$ such that $F(uv) - uv \in Z(R)$, for all $u, v \in U$, then $U \subseteq Z(R)$.

For developing the proof of the above theorem, we require the following lemmas which are essentially proved in [45].

Lemma 5.4.2. Let R be a 2-torsion free prime ring and U be a Lie ideal of R. If $U \not\subseteq Z(R)$, then $C_R(U) = Z(R)$.

Lemma 5.4.3. If $U \not\subseteq Z(R)$ is a Lie ideal of a 2-torsion free prime ring R and $a, b \in R$ such that $aUb = \{0\}$, then $a = 0$ or $b = 0$.

The following lemma is in fact, an extension of a result [104 , Lemma 2(a)] due to J. H. Mayne.

Lemma 5.4.4. Let R be a 2-torsion free prime ring and U be a Lie ideal of R such that $U \not\subseteq Z(R)$. If R admits a derivation d which is zero on U, then d is zero on R.

Proof. By our hypotheses, we have

$$d(u) = 0, \quad \text{for all } u \in U. \tag{5.4.1}$$

Replacing u by $[u, r]$ in (5.4.1), we find that $d([u, r]) = ud(r) - d(r)u = 0$ and hence $[u, d(r)] = 0$, for all $u \in U$, $r \in R$. This yields that $d(r) \in C_R(U)$. Thus, the application of Lemma 5.4.2 gives $d(r) \in Z(R)$. Hence $[d(r), s] = 0$, for all $r, s \in R$. Replacing r by rr_1 in the latter relation and using it, we obtain $d(r)[r_1, s] + [r, s]d(r_1) = 0$, for all $r, r_1, s \in R$. Now replace r_1 by $d(r)$, to get $[r, s]d^2(r) = 0$, for all $r, s \in R$. Again replacing s by us, we find that $[r, u]sd^2(r) = 0$, for all $u \in U$ and $r, s \in R$ i.e., $[r, u]Rd^2(r) = \{0\}$, for all $u \in U$, $r \in R$. Thus primeness of R implies that either $[r, u] = 0$ or $d^2(r) = 0$.

Since $U \not\subseteq Z(R)$, we have $d^2(r) = 0$, for all $r \in R$. Replace r by rs in the above relation, to get $2d(r)d(s) = 0$, for all $r, s \in R$. Since R is 2-torsion free, the latter relation yields that $d(r)d(s) = 0$, for all $r, s \in R$. We conclude that $d(r)d(sr) = \{0\}$, for all $r, s \in R$. Thus $d(r)Rd(r) = \{0\}$, for all $r \in R$. The primeness of R forces that $d = 0$. □

Proof of Theorm 5.4.10. If $F = 0$, then $uv \in Z(R)$, for all $u, v \in U$. Hence $[uv, r] = 0$, for all $u, v \in U$ and $r \in R$. This gives that $u[v, r] + [u, r]v = 0$, for all $u, v \in U$. Replacing u by $2wu$ and using the fact that $\mathrm{char}R \neq 2$, we get $[w, r]uv = 0$, for all $u, v, w \in U$ and $r \in R$. Replace r by rs, to get $[w, r]suv = 0$, for all $u, v, w \in U$ and $r, s \in R$ i.e., $[w, r]Ruv = \{0\}$, for all $u, v, w \in U$, $r \in R$. Thus primeness of R implies that either $[w, r] = 0$ or $uv = 0$. If $uv = 0$, for all $u, v \in U$, then replacing v by $[v, r]$, we get $urv = 0$, for all $u, v \in U$ and $r \in R$. Hence $uRv = \{0\}$, for all $u, v \in U$. Thus primeness of R forces that $U = \{0\}$, which is not possible. Hence we have $[w, r] = 0$, for all $w \in U$ and $r \in R$ i.e., $U \subseteq Z(R)$.

Hence onward, we assume that $F \neq 0$. Suppose on contrary that $U \not\subseteq Z(R)$. Since we have $F(uv) - uv \in Z(R)$, for all $u, v \in U$, $[F(uv) - uv, w] = 0$, for all $u, v, w \in U$. Replacing v by $2vw$ and using the fact that $\mathrm{char}R \neq 2$, we get $[(F(uv) - uv)w + uvd(w), w] = 0$, for all $u, v, w \in U$. Hence $[uvd(w), w] = 0$, for all $u, v, w \in U$ i.e.,

$$uv[d(w), w] + u[v, w]d(w) + [u, w]vd(w) = 0, \quad \text{for all } u, v, w \in U. \tag{5.4.2}$$

Replace u by $2u_1u$ in (5.4.2) and use (5.4.2), to obtain $[u_1, w]uvd(w) = 0$, for all $u, u_1, v, w \in U$. Hence $[u_1, w]Uvd(w) = \{0\}$, for all $u_1, v, w \in U$. Thus by Lemma 5.4.3, for each $w \in U$ either $[u_1, w] = 0$ or $vd(w) = 0$. Now, let

$U_1 = \{w \in U \mid vd(w) = 0, \text{ for all } v \in U\}$ and $U_2 = \{w \in U \mid [u_1, w] = 0, \text{ for all } u_1 \in U\}$. Then U_1 and U_2 both are additive subgroups of U and $U_1 \cup U_2 = U$. Thus either $U_1 = U$ or $U_2 = U$. If $U_1 = U$, then $vd(w] = 0$, for all $v, w \in U$. Replacing v by $[v, r]$ in the above relation and using it, we get $vrd(w) = 0$, for all $v, w \in U$ and $r \in R$, i.e. $URd(w) = \{0\}$, for all $w \in U$. Since R is prime and U is nonzero, we conclude that $d(w) = 0$, for all $w \in U$. Hence by Lemma 5.4.4, we get $d = 0$, a contradiction. On the other hand, if $U_2 = U$, then $[u_1, w] = 0$, for all $u_1, w \in U$. Thus by Proposition 5.2.4, we get $U \subseteq Z(R)$, again a contradiction. This completes the proof of the theorem. □

Using the similar arguments, we get the following :

Theorem 5.4.11. Let R be a 2-torsion free prime ring and U be a nonzero Lie ideal of R with $u^2 \in U$, for all $u \in U$. If R admits a generalized derivation F with associated derivation $d \neq 0$ such that $F(uv) + uv \in Z(R)$, for all $u, v \in U$, then $U \subseteq Z(R)$.

Following is the immediate consequence of Theorem 5.4.10.

Corollary 5.4.1. Let R be a prime ring. If R admits a generalized derivation F with associated derivation $d \neq 0$ such that $F(xy) - xy \in Z(R)$, for all $x, y \in R$, then R is commutative.

Remark 5.4.4. Since every ideal in a ring R is a Lie ideal of R, conclusion of the above theorem holds even if U is assumed to be an ideal of R. Though the assumption that $u^2 \in U$, for all $u \in U$ seems close to assuming that U is an ideal of the ring, but there exist Lie ideals with this property which are not ideals.

Example 5.4.7. Let $R = \left\{ \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \mid x, y, z \in \mathbb{Z} \right\}$. Then it can be easily seen that $U = \left\{ \begin{pmatrix} x & y \\ 0 & x \end{pmatrix} \mid x, y \in \mathbb{Z} \right\}$ is a Lie ideal of R satisfying $u^2 \in U$, for all $u \in U$. However, U is not an ideal of R.

Remark 5.4.5. In conclusion, it is tempting to conjecture as follows :

Conjecture 5.4.1. Let R be a 2-torsion free prime ring and U be a nonzero Lie ideal of R. If R admits a generalized derivation F with associated derivation $d \neq 0$ such that $F(uv) - uv \in Z(R)$ or $F(uv) + uv \in Z(R)$, for all $u, v \in U$, then $U \subseteq Z(R)$.

Theorem 5.4.12. Let R be a 2-torsion free prime ring and U be a nonzero Lie ideal of R with $u^2 \in U$, for all $u \in U$. If R admits a generalized derivation F with associated derivation $d \neq 0$ such that $F(uv) - vu \in Z(R)$, for all $u, v \in U$, then $U \subseteq Z(R)$.

Proof. If $F = 0$, then $vu \in Z(R)$, for all $u, v \in U$. Using the same arguments as we have used in the begining of the proof of Theorem 5.4.10, we get the required result.

Hence, onward we assume that $F \neq 0$. Suppose on contrary that $U \not\subseteq Z(R)$. Since for any $u, v \in U$, we have $F(uv) - vu \in Z(R)$, $[F(uv) - vu, v] = 0$, for all $u, v \in U$. Replacing u by $2uv$ and using the fact that char$R \neq 2$, we get $[(F(uv) - vu)v + uvd(v), v] = 0$, for all $u, v \in U$ and hence $[uvd(v), v] = 0$, for all $u, v \in U$. We have

$$uv[d(v), v] + [u, v]vd(v) = 0, \quad \text{for all } u, v \in U. \tag{5.4.3}$$

Replace u by $2wu$ in (5.4.3) and use (5.4.3), to obtain $[w,v]uvd(v) = 0$, for all $u,v,w \in U$. Hence $[w,v]Uvd(v) = \{0\}$, for all $v,w \in U$. Thus by Lemma 5.4.3, either $[w,v] = 0$ or $vd(v) = 0$. If $[w,v] = 0$, then by Proposition 5.2.4, we get $U \subseteq Z(R)$, a contradiction. On the other hand, if $vd(v) = 0$, then linearizing the above relation on v, we obtain

$$(5.4.4) \qquad ud(v) + vd(u) = 0, \quad \text{for all } u,v \in U.$$

Again replace v by $2vu$ in (5.4.4) and use the fact that char$R \neq 2$, to get $ud(vu) + vud(u) = 0$, for all $u,v \in U$. Thus (5.4.4) yields that $[u, vd(u)] = 0$, for all $u,v \in U$. This gives that $v[u,d(u)] + [u,v]d(u) = 0$, for all $u,v \in U$. Replacing v by $2wv$, we get $[u,w]vd(u) = 0$, for all $u,v,w \in U$ i.e., $[u,w]Ud(u) = \{0\}$, for all $u,w \in U$. Hence for each fixed $u \in U$, by Lemma 5.4.3 either $[u,w] = 0$ or $d(u) = 0$. Now, let $U_1 = \{u \in U \mid d(u) = 0\}$ and $U_2 = \{w \in U \mid [u,w] = 0\}$. Then U_1 and U_2 both are additive subgroups of U and $U_1 \cup U_2 = U$. Thus either $U_1 = U$ or $U_2 = U$. If $U_1 = U$, then $d(u) = 0$, for all $u \in U$ and by Lemma 5.4.4, we get $d = 0$, a contradiction. On the other hand, if $U_2 = U$, then $[u,w] = 0$, for all $u,w \in U$. Thus by Proposition 5.2.4, $U \subseteq Z(R)$, again a contradiction. Hence the result is proved. □

Using the same techniques with necessary variations we get the following :

Theorem 5.4.13. Let R be a 2-torsion free prime ring and U be a nonzero Lie ideal of R with $u^2 \in U$, for all $u \in U$. If R admits a generalized derivation F with associated derivation $d \neq 0$ such that $F(uv) + vu \in Z(R)$, for all $u,v \in U$, then $U \subseteq Z(R)$.

Remark 5.4.6. The Example 5.3.6 demonstrates that R to be prime is essential in the hypotheses of the above theorems.

BIBLIOGRAPHY

[1] Abujabal, H. A. S. : *A generalization of some commutativity theorems for rings I*, Tamkang. J. Math. 21 (3) (1990), 239-245.

[2] Abujabal, H. A. S. : *Two commutativity results for rings*, Math. Japonica 38 (2) (1993), 297-299.

[3] Abujabal, H. A. S. : *Commutativity of rings via Streb's classification*, Southeast Asian Bull. Math. 22 (1998), 1-10.

[4] Ali. Asma : *structure of certain periodic near rings*, New Zealand J. Math. 30 (2000), 1-4.

[5] Ali, Asma, Rehman, N. and Shakir Ali : *On Lie ideals with derivations as homomorphism and anti-homomorphism*, Acta. Math. Acad. Sci. Hung. 101 (2003), 79-82.

[6] Ali. Asma : *structure of certain periodic rings and near rings*, East-West J. Math. (To Appear).

[7] Argac, N., A. Kaya and A. Kisir : *(σ, τ)-derivations in prime rings*, Math. J. Okayama Univ. 29 (1987), 173-177.

[8] Ashraf, M. : *On structure and commutativity of certain periodic near rings*, Resultate Math. 24 (1993), 1-10.

[9] Ashraf, M. : *structure of certain periodic rings and near rings*, Rend. Sem. Mat. Univ. pol. Torino, 53, 1 (1995), 61-67.

[10] Ashraf, M. : *commutativity of rings with certain constraints*, Proc. R. Ir. Acad. 96 A, No. 2 (1996),177-183.

[11] Ashraf, M. : *On strong commutativity preserving traces of biadditive maps*, Riv. Mat. Univ. Parma 5 (1995), 65-70.

[12] Ashraf, M. and S. M. Al-Shammakh : *On Generalized (θ, ϕ)-derivations in rings*, Internat. J. Math Game Theory and Algebra 13 (2002).

[13] Ashraf, M. and R. Nadeem : *On commutativity of rings with derivations*, Result. Math. 42 (2001), 3-8.

[14] Ashraf, M. and R. Nadeem : *On derivations and commutativity in prime rings*, East-West J. Math. 3 (1) (2001), 87-91.

[15] Ashraf, M., R. Nadeem and M. A. Quadri : *On (σ, τ)-derivations in certain classes of rings*, Rad. Mat. 9 (1999), 187-192.

[16] Ashraf, M. and J. Vukman : *On derivations and commutativity in semiprime rings*, Aligarh Bull. Math. 18 (1999), 29-38.

[17] Ashraf, M., M. A. Quadri and Asma Ali : *Structure of certain near rings*, Rend. Instit. Math. Univ. Trieste 24 (1992), 161-167.

[18] Awtar, R. : *Lie and Jordan structure in prime rings with derivations*, Proc. Amer. Math. soc. 41 (1973), 67-74.

[19] Awtar, R. : *On a theorem of Posner*, Proc. Cambridge Philos. Soc. 73 (1973), 25-27.

[20] Awtar, R. : *Lie structure in prime rings with derivations*, Publ. Math. (Debrecen) 31 (1984), 209-215.

[21] Awtar, R. : *Lie ideals and Jordan derivations of prime rings*, Proc. Amer. Math. Soc. 90 (1984), 9-14.

[22] Bell, H. E. : *Near rings in which each element is a power of itself*, Bull. Austral. Math. Soc. 2 (1970), 363-368.

[23] Bell, H. E. : *Certain near rings are rings*, J. London Math. Soc. 4 (1971), 264-270.

[24] Bell, H. E. : *On some commutativity theorems on Herstien*, Arch. Math. 24 (1974), 25-38.

[25] Bell, H. E. : *Some commutativity results for periodic rings*, Acta. Math. Sci. Acad. Hung. 28 (1976), 279-283.

[26] Bell, H. E. : *A commutativity condition for rings*, Canad. J. Math. 28 (1976), 986-991.

[27] Bell, H. E. : *A commutativity theorem for near rings*, Canad. Math. Bull. 20 (1977), 25-28.

[28] Bell, H. E. : *A commutativity study for periodic rings*, Pacific J. Math. 70 (1977), 29-36.

[29] Bell, H. E. : *On power map and ring commutativity*, Canad. Math. Bull. 21 (1978), 399-404.

[30] Bell, H. E. : *On rings with commuting powers*, Math. Japonica 24 (1979), 437-478.

[31] Bell, H. E. : *Centres for near rings : applications to commutativity theorems* , Proc. Edinburgh Math. Soc. 23 (1980), 61-67.

[32] Bell, H. E. : *On commutativity of periodic rings and near rings*, Acta. Math. Acad. Sci. Hungar. 36 (1980), 293-302.

[33] Bell, H. E. : *On commutativity and structure of periodic rings*, Math. J. Okayama Univ. 27 (1985), 1-3.

[34] Bell, H. E. and M.N. Daif : *On commutativity and strong commutativity preserving maps*, Canad. Math. Bull. 37 (1994), 443-447.

[35] Bell, H. E. and M.N. Daif : *On derivations and commutativity in prime rings* , Acta. Math. Hung. 66 (4) (1995), 337-343.

[36] Bell, H. E. and Q. Deng : *On derivations and commutativity in semiprime rings* , Comm. Algebra 23 (1995), 3705-3713.

[37] Bell, H. E. and L. C. Kappe : *Rings in which derivations satisfy certain algebraic conditions*, Acta. Math. Hungar. 53 (1989), 339-346.

[38] Bell, H. E. and S. Ligh : *Some decomposition theorems for periodic rings and near rings*, Math. J. Okayama Univ. 31 (1989), 93-99.

[39] Bell, H. E. and J. Lucier : *On additive maps and commutativity in rings*, Resultate Math. 36 (1999), 1-8.

[40] Bell, H. E. and W. S. Martindale : *Centralizing mappings of semiprime rings*, Canad. Math. Bull. 30 (1987), 92-101.

[41] Bell, H. E. and W. S. Martindale : *Semiderivations and commutativity in prime rings*, Canad. Math. Bull. 31 (1988), 500-508.

[42] Bell, H. E. and I. Nada : *On some center-like subset of rings*, Arch. Math. 48 (1987), 381-387.

[43] Belluce L. P., I. N. Herstein and s. K. Jain : *Generalized commutative ring*, Nagoya Math. J. 27 (1966), 1-5.

[44] Bergen, J. : *Derivations in prime rings*, Canad. Math. Bull. 26 (1983), 167-170.

[45] Bergen, J., I. N. Herstein and J. W. Kerr : *Lie ideals and derivations of prime rings*, J. Algebra 71 (1981), 259-267.

[46] Bergen, J., I. N. Herstein and C, Lanski : *Derivation with invertible values*, Canad. J. Math. 35 (1983), 300-310.

[47] Brešar, M. : *On the distance of the composition of two derivations to the generalized derivations* , Glasgow Math. J. 33 (1991), 89-93.

[48] Brešar, M. : *Centralizing mappings and derivations in prime rings*, J. Algebra 156 (1993), 385-394.

[49] Brešar, M. : *Commuting traces of biadditive mappings, commutativity-preserving mappings and Lie mappings* , Trans. Amer. Math. Soc. 335 (1993), 525-546.

[50] Brešar, M. and J. Vukman : *On left derivations and related mappings*, Proc. Amer. Math. Soc. 110 (1990), 7-16.

[51] Brešar, M. and J. Vukman : *Jordan (θ, ϕ)-derivations*, Glasnik Matematicki 26 (46) (1991), 13-17.

[52] Chacron, M. : *On a theorem of Herstein*, Canad. J. Math. 21 (1969), 1348-1353.

[53] Chang, J. C. : *On fixed power central (α, β)-derivations*, Bull. Inst. Math. Acad. Sinica 15 (1987), 163-178.

[54] Clay, J. R. : *The near rings on groups of lower order*, Math. Z. 104 (1968), 364-371.

[55] Daif, M. N. : *A note on rings with derivation I, II*, Pure Math. Manuscript. 4 (1985), 87-91, 93-96.

[56] Daif, M. N. and H. E. Bell : *Remarks on derivations on semiprime rings*, Internat. J. Math. & Math. Sci. 15 (1992), 205-206.

[57] Deng, Q. : *Derivations in prime rings*, Rend. Circ. Math. Polerma 42 (1993), 21-28.

[58] Deng, Q. : (σ,τ) *derivations of prime rings*, Turkish J. Math. 18 (1994), 130-133.

[59] Deng, Q. : *A theorem on derivations in semiprime rings*, Collect. Math. 46 (1995), 315-319.

[60] Deng, Q. and M. Ashraf : *On storng commutativity preserving mappings*, Results in Math. 30 (1996), 259-263.

[61] Deng, Q. and H. E. Bell : *On derivations and commutativity in semiprime rings*, Comm. Algebra 23 (1995), 3705-3713.

[62] Faith, C. : *Algebraic division ring extensions*. Proc. Amer. Math. Soc. 110 (1960), 43-53.

[63] Felzenswalb, B. : *Derivations in prime rings*. Pro. Amer. Math. Soc. 84 (1982), 16-20.

[64] Felzenswalb, B. and A. Giambruno : *A commutativity theorem for rings with derivations*, Pacific. J. Math. 102 (1) (1982), 41-45.

[65] Frohlich, A. : *Distributively generated near rings*. Proc. Londan Math. Soc. 8 (1958),76-94.

[66] Herstein I. N. : *A generalization of theorem of Jacobson*, Amer. J. Math. 73 (1951), 756-762.

[67] Herstein I. N. : *A generalization of a theorem of Jacobson III^**. Amer. J. Math. 75 (1953), 105-111.

[68] Herstein I. N. : *A theorem on rings*, Canad. J. Math. 5 (1953), 238-241.

[69] Herstein I. N. : *The structure of certain class of rings*, Amer. J. Math. 75 (1953), 864-871.

[70] Herstein I. N. : *A note on rings with central nilpotent elements*, Proc. Amer. Math. Soc. 5 (1954), 620-629.

[71] Herstein I. N. : *An elementry proof a theorem of Jacobson*, Duke Math. J. 21 (1955), 45-48.

[72] Herstein I. N. : *Two remarks on commutativity of rings*, Canad. J. Math. 7 (1955), 411-412.

[73] Herstein I. N. : *A condition for the commutativity of rings*, Canad. J. Math. 9 (1957), 583-586.

[74] Herstein I. N. : *Power maps in rings*, Michigan Math. J. 8 (1961), 29-32.

[75] Herstein I. N. : *Lie and Jordan structures in simple, associative rings*, Pro. Amer. Math. Soc. 67 (1961), 517-531.

[76] Herstein I. N. : *Topics in ring theory*, Univ. Chicago Press, Chicago (1969).

[77] Herstein I. N. : *On the Lie structure of an associative ring*, J. Algebra 14 (1970), 561-571.

[78] Herstein I. N. : *On a result of Faith*, Canad. Math. Bull. 18 (4) (1975), 609.

[79] Herstein I. N. : *On the hypercentre of a ring*, J. Algebra 36 (1975), 151-155.

[80] Herstein I. N. : *Rings with involution*, Univ. Chicago Press, Chicago (1976).

[81] Herstein I. N. : *A commutativity Theorem*, J. Algebra 38 (1976), 112-115.

[82] Herstein I. N. : *A note on derivations*, Canad. Math. Bull. 21 (1978), 369-370.

[83] Herstein I. N. : *A note on derivations II*, Canad. Math. Bull. 22 (1979), 509-511.

[84] Hongan, M. : *A generalization of a theorem of Posner*, Math. J. Okayama Univ. 33 (1991), 97-101.

[85] Hongan, M. : *A note on semiprime rings with derivations*, Internat. J. Math. & Math. Sci. 20 (1997), 413-415.

[86] Hvala, B. : *Generalized derivations in rings*, Comm. Algebra 26 (4) (1998), 1147-1166.

[87] Jacobson, N. : *Structure theory of algebraic algebras of bounded degree*, Ann. Math. 46 (1945), 695-707.

[88] Jacobson, N. : *Structure of rings*, Amer. Math. Soc. Coll. Pub. (1956).

[89] Kaplansky, I. : *A theorem on division rings*, canad. J. Math. 3 (1951), 290-292.

[90] Kaya, K. : *On a (σ, τ)-derivations of prime rings*, Doga Tr. J. Math. D.C. 12 (2) (1988), 46-51.

[91] Komatsu, H., T. Nishinaka and H. Tominaga : *On commutativity of rings*, Rad. Math. 6 (1990), 303-311.

[92] Lambek, J. : *Lectures on rings and modules*, Blaisdell Publishing Company, Londan (1966).

[93] Lee, P. H. and T. K. Lee : *Lie ideals of prime rings with derivations*, Bull. Inst. Math. Acad. Sinica, 11 (1983), 75-80.

[94] Lee, P. H. and T. K. Lee : *Derivations centralizing symmetric or skew elements*, Bull. Inst. Math. Acad. Sinica, 14 (1986), 249-256.

[95] Lee, P. H. and T. K. Lee : *Note on nilpotent derivations*, Proc. Amer. Math. Soc. 98 (1986) 31-32.

[96] Lee, T. K. and J. S. Lin : *A result on derivations*, Proc. Amer. Math. Soc., 124 (1996), 1687-1691.

[97] Ligh, S. : *On Boolean near rings*, Bull. Austral. Math. Soc. 1 (1969), 375-379.

[98] Ligh, S. : *On distributively generated near rings*, Proc. Edinburgh Math. Soc. 16 (1969), 239-243.

[99] Ligh, S. and J. Luh. : *Some commutativity theorems for rings and near rings* Acta. Math. Acad. Sci Hungar 28 (1976), 19-23.

[100] Ligh, S. and J. Luh : *Direct sum of J-rings and zero rings*, Amer. Math. Monthly 96 (1989), 40-41.

[101] Lithman, I. : *Rings that are radical over a commutative subring*, Math. shornik 83 (1970), 513-523.

[102] Mayne J. H. : *Centralizing automorphisms of prime rings*, Canad. Math. Bull. 19 (1976), 113-115.

[103] Mayne J. H. : *Ideals and centralizing mappings in prime rings*, Proc. Amer. Math. Soc. 86 (1982), 211-212.

[104] Mayne J. H. : *Centralizing mappings of prime rings*, Canad. Math. Bull. 27 (1984), 122-126.

[105] Mayne J. H. : *Centralizing automorphisms of Lie ideals in prime rings*, Canad. Math. Bull. 35 (1992), 510-514.

[106] McCoy, N. H. : *Rings and Ideals*, Math. Assoc. Amer. Maryland, 1948.

[107] McCoy, N. H. : *The Theory of Rings*, Macmillan, New York, 1967.

[108] Neumann, B. H. : *On the commutativity of addition*, J. Londan Math. Soc., 15 (1962), 203-208.

[109] Pilz, G. : *Near rings*, North Holland, Amsterdam, (1977).

[110] Posner, E. C. : *Derivations in prime rings*, Proc. Amer. Math. Soc., 8 (1957), 1093-1100.

[111] Quadri, M. A., M. Ashraf and Asma Ali : *Certain conditions under which near rings are rings*, Bull. Austral. Math. Soc. 42 (1990), 91-94.

[112] Quadri, M. A., M. Ashraf and Asma Ali : *Certain conditions under which near rings are rings - II*, Rad. Mat. 8 (1992), 311-319.

[113] Quadri, M. A., M. Ashraf and Asma Ali : *Structure of certain near rings*, Rend. Instit. Math. Univ. Trieste 24 (1992), 161-167.

[114] Quadri, M. A., Asma Ali and Achlesh Kumari : *Structure of rings with some polynomial constraints*, Indian J. Math. 40 (1998), 1-5.

[115] Reddy, Y. F. and C. V. L. N. Murty : *Semi-symmetric ideals in near rings*, Indian J. Pure Appl. Math. 16 (1985), 17-21.

[116] Rehman, N. : *On commutativity of rings with generalized derivations*, Math. J. Okayama Univ. 44 (2002), 43-49.

[117] Searcoid. M. O. and D. MacHale : *Two elementry generalizations of Boolean rings*, Amer. Math. Monthly 93 (1986), 121-122.

[118] Streb, W. : *Zur struktur nichtkommutativer ringe*, Math. J. Okayama Univ. 31 (1989), 135-140.

[119] Tominaga, H. and A. Yaqub : *Commutativity theorems for rings with constraints involving a commutative subset*, Resultate Math. 11 (1987), 186-192.

[120] Tong, J. : *On the commutativity of a rings with identity*, Canad. Math. Bull. 72 (1984), 456-460.

[121] Vukman, J. : *Derivation of prime rings*, Obzornic Mat. Fiz. 41 (1994), 8-14.

[122] Vukman, J. : *Derivations on semiprime rings*, Bull. Austral. Math. Soc. 53 (1995), 353-359.

[123] Vukman, J. : *Centralizers on prime and semiprime rings*. Comment Math. Univ. Carolinae, 38 (1997), 231-240.

[124] Wedderburn, J. H. M. : *A theorem on finite algebras*, Trans. Amer. Math. Soc. 6 (1905), 349-352.